INDEX

Name SUNAKAR SINGH. Subject E. MATH-I (TH-3)

STD. ______ Div. ______ Roll No. ______ School PNS.

1: BRIDGE COURSE:

A: BASIC TERMINOLOGIES:

Give basic ideas about following terms:

Angle – କୋଣ
Arc – ଚାପ
Circle – ବୃତ୍ତ
Radius – ବ୍ୟାସାର୍ଦ୍ଧ
Diameter – ବ୍ୟାସ
Hypotenuse – କର୍ଣ୍ଣ
Diagonal – କର୍ଣ୍ଣ
Height – ଉଚ୍ଚତା
Base – ଭୂମି
Perpendicular – ଲମ୍ବ.
Point – ବିନ୍ଦୁ
Co-linear points – ଏକରେଖୀୟ ବିନ୍ଦୁ
Non co-linear points – ଅଣରେଖୀୟ ବିନ୍ଦୁ.
Straight line – ସରଳ ରେଖା / ରେଖା
Segment – ରେଖା ଖଣ୍ଡ
Ray – ରଶ୍ମି
Centre – କେନ୍ଦ୍ର
Circumference – ପରିଧି
Triangle – ତ୍ରିଭୁଜ
Median – ମଧ୍ୟମା
Centroid – ଭରକେନ୍ଦ୍ର
Sphere – ଗୋଲକ
Plane – ସମତଳ
Rectangle – ଆୟତ କ୍ଷେତ୍ର
Square – ବର୍ଗ ଚିତ୍ର
Parallelogram – ସାମାନ୍ତରିକ କ୍ଷେତ୍ର
Rhombus – ରମ୍ବସ.
Cube – ସମଘନ
Cuboid – ଘନାକୃତି ବସ୍ତୁ
Equation – ସମୀକରଣ
Linear equation – ଏକଘାତୀ ସମୀକରଣ
Quadratic equation – ଦ୍ୱିଘାତୀ ସମୀକରଣ

Polynomial – ବହୁପଦ ବିଶିଷ୍ଟ
Monomial – ଏକପଦ ବିଶିଷ୍ଟ ବୀଜଗାଣିତିକ ରାଶି.
Binomial – ଦ୍ୱିପଦ ବିଶିଷ୍ଟ ରାଶି.
Root – ମୂଳ / ବୀଜ / ସମାଧାନ
Length – ଦୈର୍ଘ୍ୟ
Breadth – ପ୍ରସ୍ଥ
Thickness – ମୋଟେଇ
Area – କ୍ଷେତ୍ରଫଳ
Volume – ଆୟତନ / ଘନଫଳ
Perimeter – ପରିସୀମା
Sum/Addition – ଯୋଗ
Difference/Substraction – ବିୟୋଗ
Multiplication/Product – ଗୁଣନ
Division – ହରଣ
Dividend – ଭାଜ୍ୟ
DIVISOR – ଭାଜକ
Quotient – ଭାଗଫଳ
Remainder – ଭାଗଶେଷ
Quadrilateral – ଚତୁର୍ଭୁଜ.
Scalene triangle – ବିଷମବାହୁ Δ
Isoscales " – ସମଦ୍ୱିବାହୁ Δ
Equilateral Triangle – ସମବାହୁ Δ
Right-angled Triangle – ସମକୋଣୀ Δ
acute angle – ସୂକ୍ଷ୍ମ କୋଣ.
obtuse angle – ସ୍ଥୂଳ କୋଣ

Right angle - ସମକୋଣ (90°)

Straight angle - ସରଳ କୋଣ (180°)

Zero angle - ଶୂନ କୋଣ (0°)

Complete angle - ସଂପୂର୍ଣ୍ଣ କୋଣ (360°)

Reflex angle - ପ୍ରବୃଦ୍ଧ କୋଣ

Incentre - ଅନ୍ତଃ କେନ୍ଦ୍ର

Circumcentre - ପରି କେନ୍ଦ୍ର

Incircle - ଅନ୍ତଃ ବୃତ୍ତ.

Circumcircle - ପରିବୃତ୍ତ

Trapezium - ଟ୍ରାପିଜିୟମ୍.

B: ALGEBRAIC FORMULAE:

i) $(a+b)^2 = a^2+2ab+b^2$

ii) $(a-b)^2 = a^2-2ab+b^2$

iii) $a^2+b^2 = (a+b)^2-2ab$

(iv) $a^2+b^2 = (a-b)^2+2ab$

(v) $(a+b)^2+(a-b)^2 = 2(a^2+b^2)$

vi) $(a+b)^2-(a-b)^2 = 4ab$

(vii) $(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+b^3+3ab(a+b)$

viii) $(a-b)^3 = a^3-3a^2b+3ab^2-b^3 = a^3-b^3-3ab(a-b)$

ix) $a^3+b^3 = (a+b)^3-3ab(a+b) = (a+b)(a^2-ab+b^2)$

(x) $a^3-b^3 = (a-b)^3+3ab(a-b) = (a-b)(a^2+ab+b^2)$

xi) $a^4+a^2b^2+b^4 = (a^2+ab+b^2)(a^2-ab+b^2)$

xii) $a^2-b^2 = (a+b)(a-b)$

xiii) $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

(xiv) $(x+a)(x+b) = x^2+(a+b)x+ab$

(xv) $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= \frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$.

(C) SOLUTION OF SIMULTANEOUS LINEAR EQUATION (TWO VARIABLE)

→ General form: $a_1x+b_1y+c_1=0$ —— (1)

$a_2x+b_2y+c_2=0$ —— (2)

→ METHODS: Substitution

ELIMINATION

CROSS MULTIPLICATION

GRAPHICAL METHOD

CRAMER'S RULE.

→ Cross multiplication formula:

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

D: QUADRATIC EQUATION:

→ General form: $ax^2+bx+c=0,\ a\neq 0$

→ Solution: $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

→ Discriminant: $D = b^2-4ac$.

→ Sum of roots $= -\frac{b}{a}$, product of roots $= \frac{c}{a}$

→ $D>0$ & D is perfect square ⇒ roots are real, distinct, rational

$D>0$ & D is not perfect square ⇒ roots are real, distinct, irrational

$D<0$ ⇒ roots are complex conjugate.

$D=0$ ⇒ roots are real and equal.

E: EUCLIDEAN ALGORITHIM:

Dividend = Divisor × Quotient + Remainder.

F: REMAINDER THEOREM:

For a polynomial $P(x)$,

i) If $P(a)=0 \Leftrightarrow x-a$ is a factor of $P(x)$

ii) If $P(\frac{a}{k})=0 \Leftrightarrow kx-a$ is a factor of $P(x)$.

G: CONCEPTS OF POLYNOMIAL AND FACTORISATION:

→ using Algebraic formulae

→ using Middle term factorisation

→ using remainder theorem.

H: LAWS OF INDICES:

i) $x^m \times x^n = x^{m+n}$

ii) $\frac{x^m}{x^n} = x^{m-n}$

(iii) $(x^m)^n = (x^n)^m = x^{mn}$

(iv) $(x\times y)^m = x^m \times y^m$

(v) $\left(\frac{x}{y}\right)^m = \frac{x^m}{x^n}$

vi) $x^0 = 1,\ x\neq 0$

vii) $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

viii) $x^{-n} = \frac{1}{x^n}$

(ix) $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

I: LOGARITHIM:

→ If $a^x = y \Leftrightarrow \log_a y = x$, $a, x, y \in R,\ a\ngtr 0,\ a\neq 1,\ y>0$

→ $\log_a 1 = 0$

→ $\log_e x = \log x = \ln x$

→ $\log_a a = 1$ → $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

→ $\log_a(xy) = \log_a x + \log_a y$ → $\log_a x^n = n\log_a x$ → $\log_y x \times \log_x y = 1$

J: SET AND NUMBER SYSTEM:

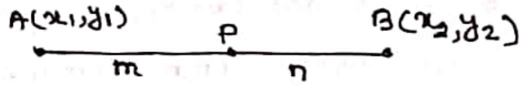
* Set ?
* Empty set * Subset & Superset.
* Finite set & infinite set
* UNION, intersection, Difference, symmetric difference.
* N, N^*, Z, Q, Q' & R.

K: CO-ORDINATE GEOMETRY:

* Introduction to cartesian co-ordinate system
* $A(x_1, y_1)$ & $B(x_2, y_2)$

$$AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

* → P, Divides $\overline{AB}$ internally in the ratio m:n



$$\therefore P = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

* → If P, Divides $\overline{AB}$ externally in the ratio m:n

$$\therefore P = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$

→ If P is the mid-point $\Rightarrow P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

→ If P Divides $\overline{AB}$ in K:1 ratio

$$P = \left(\frac{Kx_2+x_1}{K+1}, \frac{Ky_2+y_1}{K+1}\right),$$

when $K > 0 \Rightarrow$ internally

$K < 0 \Rightarrow$ externally.

L: TRIGONOMETRY:

* T-ratios in terms of P, b, h
* Addition, substraction, Multiplication and Division
* Reciprocal relation
* Division relation
* identities Relation
* Trigonometric table, * ASTC Rule
* T-ratios of compound Angle
* $\sin C \pm \sin D$, $\cos C \pm \cos D$, $\sin(A+B)+\sin(A-B)$
  $\sin(A+B).\sin(A-B)$ - - - - .
* 2A & 3A formula.

M: RELATION & FUNCTION:

→ If A and B are any two non empty sets, then any subset of $A \times B$ is called a Relation from A to B i.e if $R \subset A \times B$ then R is called a relation from A to B

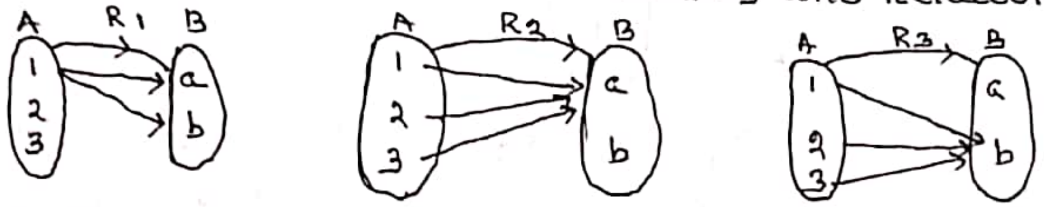
→ We know that $\phi \subset A \times B$ and $A \times B \subset A \times B$. Hence ϕ and $A \times B$ are also relations from A to B. ϕ is the smallest relation and $A \times B$ is the Largest relation from A to B.

→ Example-1: Let $A = \{1, 2, 3\}$, $B = \{a, b\}$

then $A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

Let $R_1 = \{(1,a), (1,b)\}$, $R_2 = \{(1,a), (2,a), (3,a)\}$, $R_3 = \{(2,b), (3,b), (1,b)\}$

Here $R_1, R_2, R_3 \subset A \times B$. Hence R_1, R_2, R_3 are relations from A to B.



→ If R is a relation from A to B then

Domain of $R = \text{dom}.R = D_R = \{x \mid (x,y) \in R\}$

Range of $R = \text{Rng}.R = R_R = \{y \mid (x,y) \in R\}$

→ In above examples:

$D_{R_1} = \{1\}$, $R_{R_1} = \{a, b\}$

$D_{R_2} = \{1,2,3\}$, $R_{R_2} = \{a\}$

$D_{R_3} = \{1,2,3\}$, $R_{R_3} = \{b\}$

N: FUNCTION:

→ A Relation f from X to Y is said to be a function if (a) $D_f = X$

(b) f is not one many.

→ Hence every function must be a Relation but the converse is not necessarily true.

→ If f is a function from X into Y such that $x \in X$ is Related to $y \in Y$ then we write it as $f : X \to Y$ s.t $f(x) = y$ or $(x,y) \in f$ or xfy.

→ Here y = image of x,

x = pre-image of y.

X = Domain of $f = D_f$

Y = Co-domain of f = Co.dom.f.

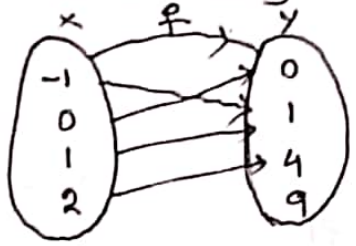
→ Range of f is $R_f = \{y \in Y \mid y = f(x), x \in X\}$.

clearly $R_f \subseteq Y$.

→ Example-2: Let $X = \{-1, 0, 1, 2\}$, $Y = \{0, 1, 4, 9\}$

$f = \{(-1,1), (0,0), (1,1), (2,4)\}$

Here f is a function from X into Y for f is not one-many & $D_f = X$



$D_f = \{-1, 0, 1, 2\} = X$, $R_f = \{0, 1, 4\}$, $\text{Codom} f = Y$

clearly for $x \in X$, $\exists\, y \in Y$ s.t $y = f(x) = x^2$.

In Ex-1, R_1 is not a function, But R_2, R_3 are functions

→ If $f(x)$ is a function then $f(a)$ is called functional value of $f(x)$ at $x = a$

→ Ex-3: Let $f(x) = x^2+1$

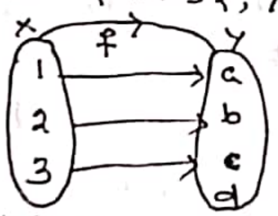
Functional value of $f(x)$ at $x=1$ is $f(1) = 1^2+1 = 2$.

0: TYPES OF FUNCTION:

0_1: Into function:

→ A function $f: X \to Y$ is said to be into function if there exist atleast one element in Y which has no preimage in X.
i.e R_f is a proper subset of Y.

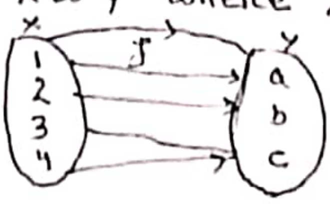
→ $f = \{(1,a), (2,b), (3,c)\}$. is an into function from X to Y where $X = \{1, 2, 3\}$, $Y = \{a, b, c, d\}$



0_2: onto function:

→ A function $f: X \to Y$ is said to be onto if $R_f = Y$

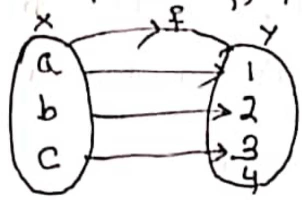
→ $f = \{(1,a), (2,b), (3,c), (4,c)\}$ is onto function from X to Y where $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c\}$



O_3: one-one function:

→ A function $f: X \to Y$ is said to be one-one if each distinct element in X has distinct image in Y. i.e if $x_1, x_2 \in X$ s.t $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

→ $f = \{(a,1),(b,2),(c,3)\}$ is an one-one function from X to Y where $X = \{a,b,c\}$, $Y = \{1,2,3,4\}$.



O_4: Many-one function

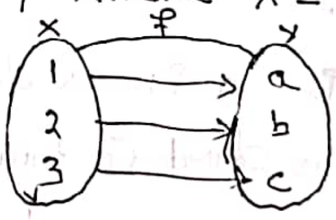
→ A function $f: X \to Y$ is said to be many-one function if there exist atleast one element in Y which has more than one preimage in X.

→ function f in O_2 is a many-one function.

O_5: Bijective function:

→ A function $f: X \to Y$ is said to be bijective if f is both one-one and onto i.e for each distinct element in X has distinct image in Y and every element of Y has distinct pre image in X

→ $f = \{(1,a),(2,b),(3,c)\}$ is a bijective function from X to Y where $X = \{1,2,3\}$, $Y = \{a,b,c\}$



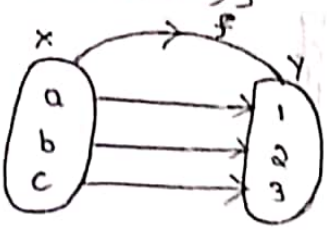
O_6 :- INVERSE OF A FUNCTION:

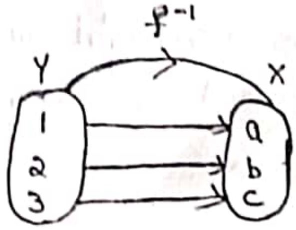
→ If $f: X \to Y$ is a bijective function then inverse of f is denoted by f^{-1} and defined by $f^{-1}: Y \to X$ s.t

$$y = f(x) \Rightarrow x = f^{-1}(y) \quad \forall x \in X, y \in Y$$

$f = \{(a,1)(b,2)(c,3)\}$ is bijective

$f^{-1} = \{(a,1).(2,b),(3.c)\}$





O_7: Composition of two function :

If $f: X \to Y$ and $g: Y \to Z$ be any two functions then Composition of f and g is denoted by gof and defined by $(gof): X \to Z$ s.t $(gof)(x) = g(f(x))$, $\forall\, x \in X$.

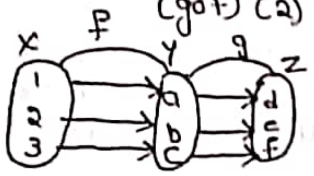
Example : Let $f = \{(1,a),(2,b),(3,c)\}$ is a function from X to Y

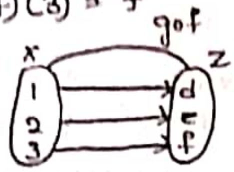
and $g = \{(a,d),(b,e),(c,f)\}$ is a function from Y to Z

then $(gof) = \{(1,d),(2,e),(3,f)\}$ is a function from X to Z

for $(gof)(1) = g(f(1)) = g(a) = d$

$(gof)(2) = e$, $(gof)(3) = f$





Example:

Let $f(x) = \sin x$ and $g(x) = x^2$

Now $(fog)(x) = f(g(x)) = f(x^2) = \sin x^2$

$\therefore$ $\sin x^2$ is the Composition of x^2 and $\sin x$.

O_8 : Real valued Function :

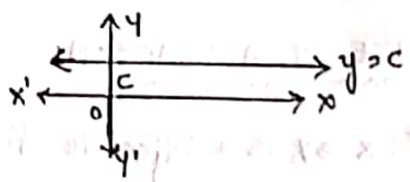
A function $f: X \to Y$ is Called a real valued function If $D_f = X \subset R$ and $Y \subset R$.

O_9 : CONSTANT FUNCTION :

The function $f: R \to R$ defined by $f(x) = c$, $\forall\, x \in R$, $c \in R$ is a Constant, is Called Constant function

→ $D_f = R$, $R_f = \{c\}$

→ graph is parallel to x-axis.

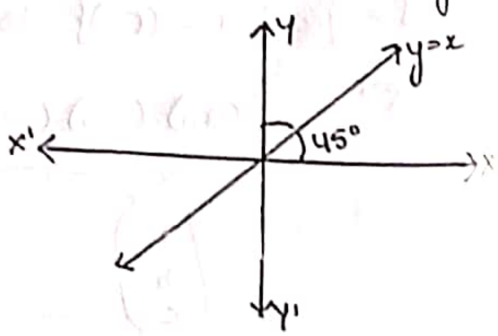


O_{10}: Identity Function :

→ A function $f: R \to R$ defined by $f(x) = x$, $\forall\, x \in R$ is Called identity function

→ $D_f = R$, $R_f = R$

→ Its graph is a straight line passing through origin and bisecting the angle between the axes.

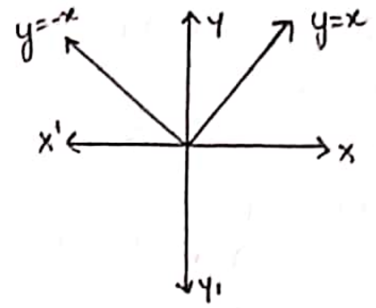


a_{11}: Absolute value / Modulus function

→ The function $f : R \to R$ defined by $\boxed{f(x) = |x| = x, x \geq 0 \\ = -x, x < 0}$ is called absolute value function

→ $D_f = R$, $R_f = R^+ = [0, \infty]$

→ Its graph is given in fig



a_{12}: Signum Function: A function

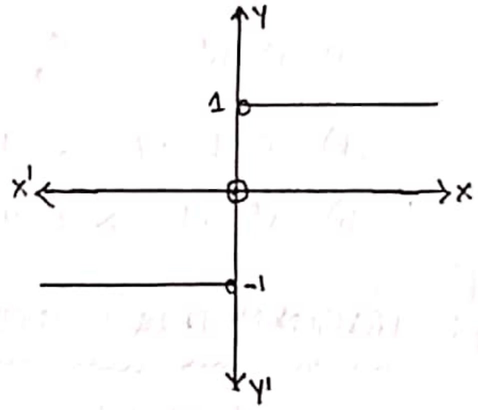
→ $f : R \to R$ defined by

$$\boxed{f(x) = sgn(x) = \frac{|x|}{x}, x \neq 0 \\ 0, x = 0}$$

is called Signum function

→ $D_f = R$, $R_f = \{-1, 0, 1\}$

→ graph is given in fig



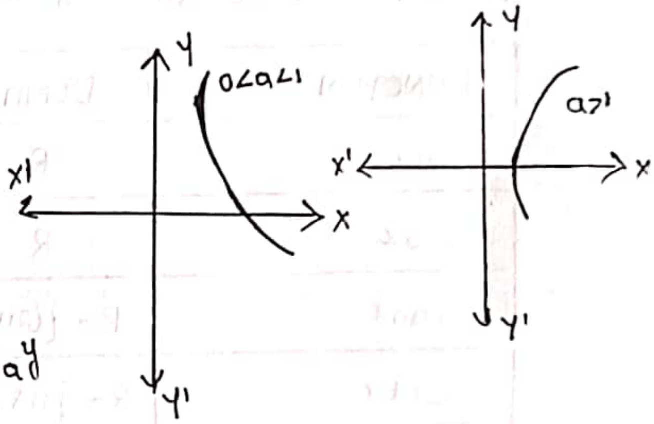
a_{13}: Logarithim function:

→ The function $f : R \to R$ defined by

If $a^{f(x)} = x \Leftrightarrow f(x) = \log_a x$.

Where $x > 0, a > 0, a \neq 1, \forall x, a \in R$. is called logarithim function

→ $D_f = (0, \infty)$, $R_f = R$.



→ Note that (a) $\log_a(xy) = \log_a x + \log_a y$

(b) $\log_a \left[\frac{x}{y}\right] = \log_a x - \log_a y$

(c) $\log_a x^n = n \log_a x$

(d) $\log_a a = 1$

(e) $\log_a x = 0 \Leftrightarrow x = 1$ ie $\log_a 1 = 0$

(f) $\log_a x = \frac{1}{\log_x a}$

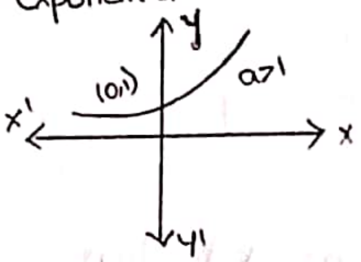
(g) $\log_a x = \log_b x \times \log_a b$

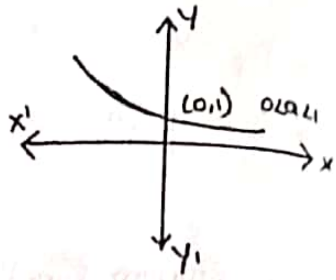
(h) $e^{\ln x} = x$, $a^{\log_a x} = x$ (i) $\log_e x = \log x = \ln x$

0.14: EXPONENTIAL FUNCTION

→ The function $f: R \to R$ defined by

$f(x) = a^x$, $a > 0$, $a \neq 1$, $a, x \in R$ is called exponential function





→ $D_f = R$, $R_f = (0, \infty)$

→ Note that

(a) $a^x . a^y = a^{x+y}$

(b) $\frac{a^x}{a^y} = a^{x-y}$, (c) $(a^x)^y = (a^y)^x = a^{xy}$

(d) $a^x \times b^x = [a \times b]^x$, (e) $\frac{a^x}{b^x} = \left[\frac{a}{b}\right]^x$

(f) $a^0 = 1, a \neq 0$, (g) $x^{-n} = \frac{1}{x^n}$, $\left[\frac{x}{y}\right]^{-n} = \left[\frac{y}{x}\right]^n$

(h) $a^x = a^y \iff x = y, a \neq 1$, (i) $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

0.15: TRIGONOMETRIC FUNCTION:

sin x, cos x, tan x, cot x, sec x and cosec x are called trigonometric functions. These are real valued functions. x is measured in radian

FUNCTION	DOMAIN	RANGE
sin x	R	$[-1, 1]$
cos x	R	$[-1, 1]$
tan x	$R - \{(2n+1)\frac{\pi}{2} : n \in z\}$	R
cot x	$R - \{n\pi : n \in z\}$	R
sec x	$R - [(2n+1)\frac{\pi}{2} : n \in z$	$R - (-1, 1)$
cosec x	$R - [n\pi : n \in z]$	$R - (-1, 1)$

016: INVERSE TRIGONOMETRIC FUNCTION

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$ are called inverse trignometric function which you have known

FUNCTION	DOMAIN	RANGE
$\sin^{-1}x$	$[-1,1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot^{-1}x$	R	$[0,\pi]$
$\sec^{-1}x$	$[-\infty,-1] \cup [1,\infty]$	$[0,\pi] - \{\frac{\pi}{2}\}$
$\text{cosec}^{-1}x$	$[-\infty,-1] \cup [1,\infty]$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

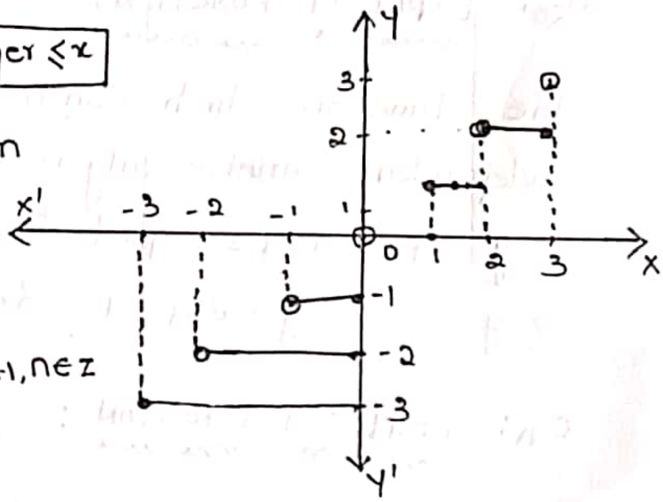
017: Greatest integer function (Bracket function)

→ For all $x \in R$, $\boxed{[x] = \text{Greatest integer} \leq x}$ is called Greatest integer function

→ $D_f = R$, $R_f = Z$

→ Thus

$$[x] = \begin{cases} x & \text{if } x \in Z \\ n & \text{if } x \notin Z \text{ and } n < x < n+1, n \in Z \end{cases}$$



→ Examples:

$[0] = 0$, $[1] = 1$, $[2] = 2$, $[-1] = -1$

$[-3] = -3$, $[-2] = -2$

$[2.5] = 2$ for $2 < 2.5 < 3$

$[-2.5] = -3$ for $-3 < -2.5 < -2$.

$[\sqrt{3}] = 1$ for $1 < \sqrt{3} < 2$

$[e] = 2$ for $2 < e < 3$

$[-e] = -3$ for $-3 < -e < -2$

$[\pi] = 3$, $[-\pi] = -4$.

Q_{18}: ALGEBRAIC FUNCTION

There are three types of algebraic function

(i) Polynomial function: $p(x) = a_0 + a_1 x + a_2 x^2 + \dots\dots + a_n x^n$

Ex x^2+2x+3, $3x+5e$

(ii) Rational function $\frac{P(x)}{Q(x)} = \frac{a_0+a_1x+a_2x^2+\dots\dots+a_nx^n}{b_0+b_1x+b_2x^2+\dots\dots+b_nx^n}$

Ex $\frac{x}{x^2+1}$, $\frac{x^2+2x+5}{3x+1}$

(iii) Irrational function: $p(x)^{\frac{p}{q}}$

Ex: $\sqrt{x}$, $(x^2+2x+1)^{2/3}$

Q_{19}: TRANSENDENTAL FUNCTION:

Trigonometric, logarithmic, exponential function, inverse trigonometric functions are Called transcendental function.

Q_{20}: EXPLICIT FUNCTION:

The function which Can be expressed directly in terms of independent variable only is Called explicit function

→ $y = f(x)$ is the explict function

→ $y = \sin x$, $y = x^2+1$ etc. are explicit function.

Q_{21}: IMPLICIT FUNCTION:

→ The function in the form $f(x,y)=0$ i.e in which x and y Can't be Separated from Each other is Called implicit function

→ Ex: $x^2+y^2=4$, $x^3+y^3-3axy=0$

→ An implicit function Can be Converted into explicit function whenever possible

Ex: $x^2+y^2=4$ (implicit)

$\Rightarrow y = \sqrt{4-x^2}$ (Explicit)

Q_{22}: Even function

→ A function $f(x)$ is said to be an even function if

$$\boxed{f(-x) = f(x) \; \forall x}$$

→ $\cos x$ is an even function

for $f(x) = \cos x$

$\Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$ $(\because \cos(-\theta) = \cos\theta)$

Similarly $\sec x$, x^2, x^4 are even functions

Q_{23}: Odd FUNCTION:

→ A Function $f(x)$ is said to be an odd function if

$$\boxed{f(-x) = -f(x) \; \forall x}$$

→ $\sin x$ is an odd function for

$f(x) = \sin x \Rightarrow f(-x) = \sin(-x) = -\sin x = -f(x)$

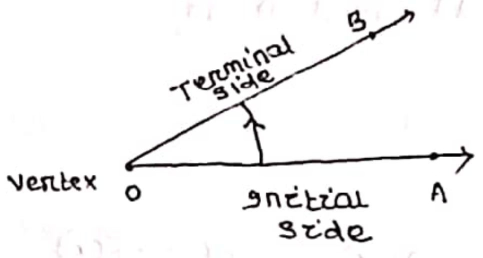
Similarly $\tan x$, $\cot x$, $\operatorname{cosec} x$, x^3 are odd function

A: CONCEPT OF ANGLE:

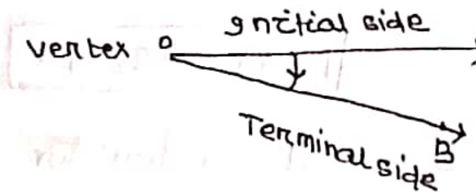
→ An Angle is a geometric figure obtained by rotating a given ray about its initial point.

→ The original ray is called initial side (line), the final ray is called terminal side and the point of rotation is called vertex.

→ If the direction of rotation is anticlockwise then the angle is called positive (+ve) angle and if the direction of rotation is clockwise then the angle is called negative angle (-ve).



(+ve angle)



(-ve angle)

A1: Types of Angle: (θ):

Zero angle → $\theta = 0°$

Acute angle → $0° < \theta < 90°$

Right angle → $\theta = 90°$

Obtuse angle → $90° < \theta < 180°$

Straight angle → $\theta = 180°$

Reflex angle → $180° < \theta < 360°$

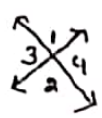
Complete angle/full rotation → $\theta = 360°$

Adjacent angles → Angle with common side, common vertex and their interior are disjoint.

Complementary Angles → sum of two angles

Supplementary Angles → sum of measures of two angles $= 180°$

Vertical Angles → $\angle 1$ & $\angle 2$, $\angle 3$ & $\angle 4$



A2: SYSTEM OF MEASUREMENT OF ANGLE

→ There are three system of measurements of angle.

→ SEXAGESIMAL / ENGLISH SYSTEM:

1 right angle $= 90°$ (Degree) ——(1)

$1° = 60'$ (minutes), $1' = 60''$ (second).

→ CENTISIMAL SYSTEM:

1 right angle $= 100^G$ (grade) —— (2)

$1^G = 100'$

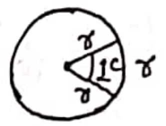
$1' = 100''$

→ CIRCULAR SYSTEM:

→ In this system the unit of angle is Radian (c/R)

→ A radian is the angle substended at the centre of a circle by an arc whose length is equal to radius of the circle.

Hence $\pi^c = 180°$ —— (3)



→ From eqn (1), (2), (3)

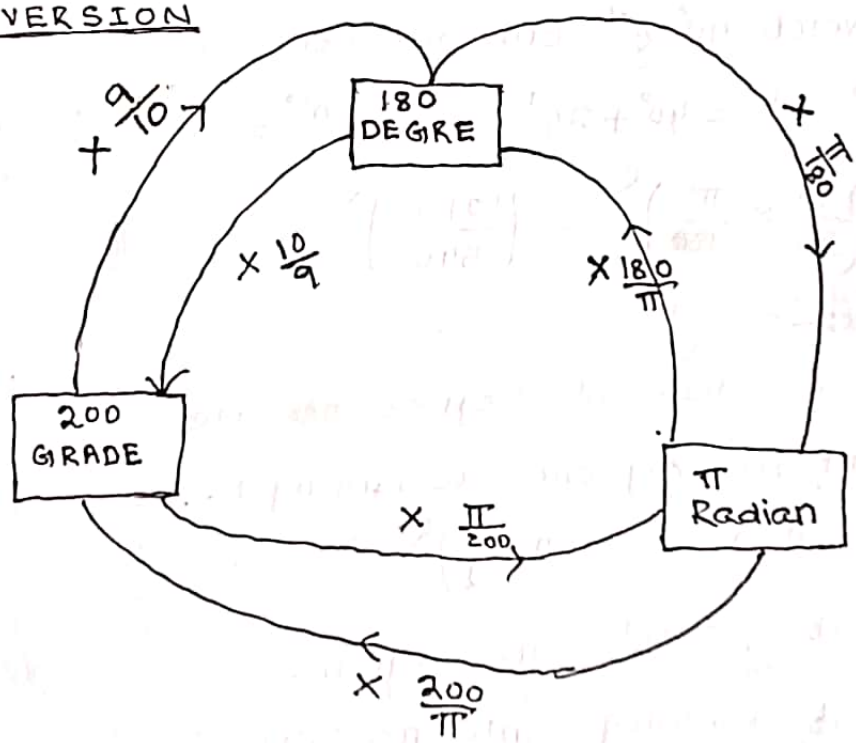
$$\boxed{\pi^c = 180° = 200^G}$$

→ There are three units of measurement of Angle. Degree, Radian, Grade.

→ Relation: Degree ≡ D, Radian ≡ R, Grade ≡ G then

$$\boxed{\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}}$$

→ CONVERSION



→ Note that 1 radian $= 57° 17' 44.8'' \approx 57° 17' 45''$

Ex-1 Convert the following into remaining two units of measurement of an angle.

(i) 30° (ii) 2^G (iii) $\left(\frac{\pi}{3}\right)^R$.

(i) $30^\circ = \left(30 \times \frac{10}{9}\right)^G = \left(\frac{100}{3}\right)^G$

$30^\circ = \left(30 \times \frac{\pi}{180}\right)^R = \left(\frac{\pi}{6}\right)^R$

(ii) $2^G = \left(2 \times \frac{9}{10}\right)^\circ = \left(\frac{9}{5}\right)^\circ = (1 \cdot 8)^\circ = 1^\circ + (0 \cdot 8)^\circ$

$= 1^\circ + (0.8 \times 60)' = 1^\circ\, 48'$

$2^G = \left(2 \times \frac{\pi}{200}\right)^R = \left(\frac{\pi}{100}\right)^R$

(iii) $\left(\frac{\pi}{3}\right)^R = \left(\frac{\pi}{3} \times \frac{180}{\pi}\right)^\circ = 60^\circ$

$\left(\frac{\pi}{3}\right)^R = \left(\frac{\pi}{3} \times \frac{200}{\pi}\right)^G = \left(\frac{200}{3}\right)^G$.

Ex-2 Convert $40^\circ\, 20'$ into radian.

Sol^n $40^\circ\, 20' = 40^\circ + 20' = 40^\circ + \frac{20}{60}^\circ = 40^\circ + \frac{1}{3}^\circ = \left(\frac{121}{3}\right)^\circ$

$= \left(\frac{121}{3} \times \frac{\pi}{180}\right)^R = \left(\frac{121\pi}{540}\right)^R$.

HOME TASK:-1

① Convert 6 radian into Degree measure

(2) Convert following into remaining two units of measurement

(i) 45° (ii) 5^G (iii) $\left(\frac{\pi}{2}\right)^R$.

(3) Convert $45^\circ\, 30'$ into radian.

(4) Convert following into radian

(i) 660° (ii) -270° (iii) 1440° (iv) 0°

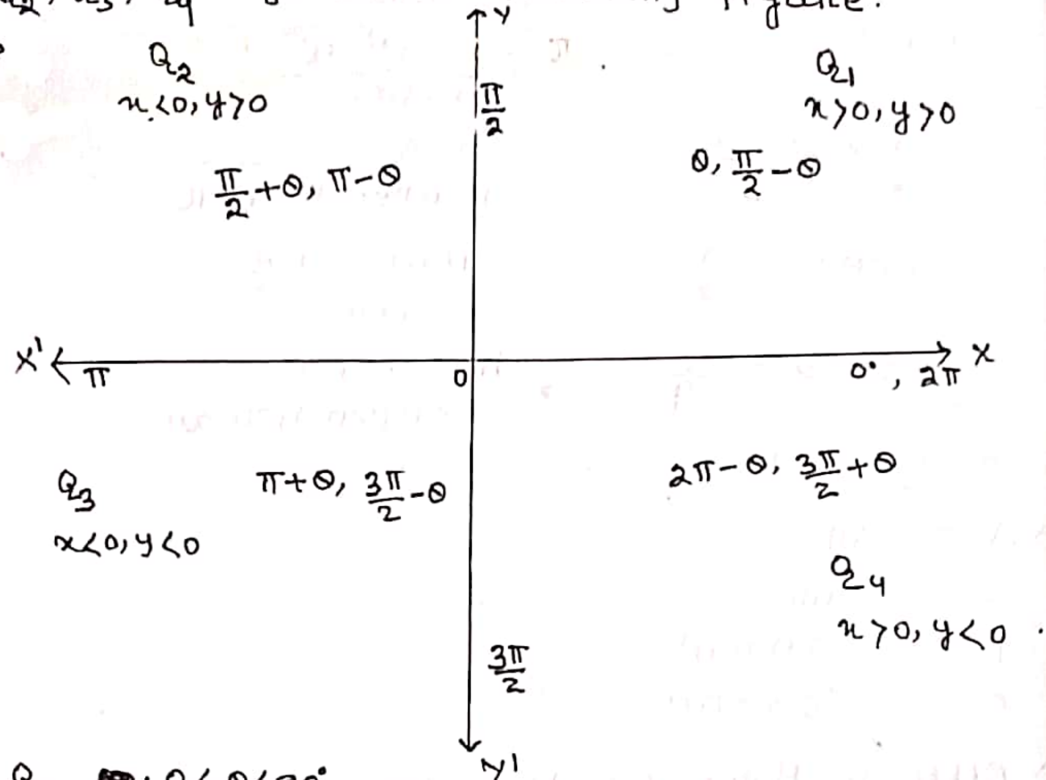
(5) Convert following into degrees

(i) 17π (ii) $\frac{9\pi}{2}$ (iii) $\frac{82\pi}{6}$ (iv) $\frac{\pi}{3}$

B: QUADRANTS:

→ The co-ordinate axes (X- & Y-axis) divide the plane into four infinite regions called Quadrants denoted by Q_1, Q_2, Q_3, Q_4 given in following figure.

→ If $\theta < 90°$



→ In Q_1, $0 < \theta < 90°$

In Q_2, $90° < \theta < 180°$

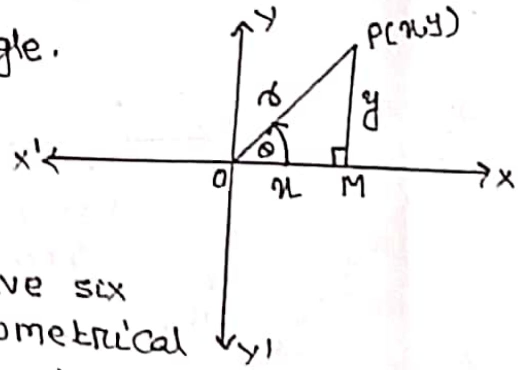
Q_3, $180 < \theta < 270°$

Q_4, $270 < \theta < 360°$.

C: TRIGONOMETRIC RATIOS (T-ratios)

→ Consider a line OP making an Angle θ with +ve direction of X-axis measured in anticlockwise direction.

Let $PM \perp$ X-axis. P be the point (x, y). Then $OM = x$, $PM = y$. Let $OP = r$ then $r = \sqrt{x^2+y^2}$.



Then OMP is a right angled triangle.

PM = Perpendicular

OM = base

OP = Hypotenuse.

→ Using these three sides we have six ratios which are called trigonometrical ratios. These are sine ($\sin\theta$), cosine ($\cos\theta$), tangent ($\tan\theta$), co-tangent ($\cot$), Secant ($\sec$.) and Cosecant (cosec) which are defined below:

$\rightarrow$ $\sin\theta = \frac{y}{r} = \frac{\text{perpendicular}}{\text{hypotenuse}}$.

$\cos\theta = \frac{x}{r} = \frac{\text{base}}{\text{hypotenuse}}$

$\tan\theta = \frac{y}{x} = \frac{\text{perpendicular}}{\text{base}}$.

$\cot\theta = \frac{x}{y} = \frac{\text{base}}{\text{perpendicular}}$

$\sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{base}}$.

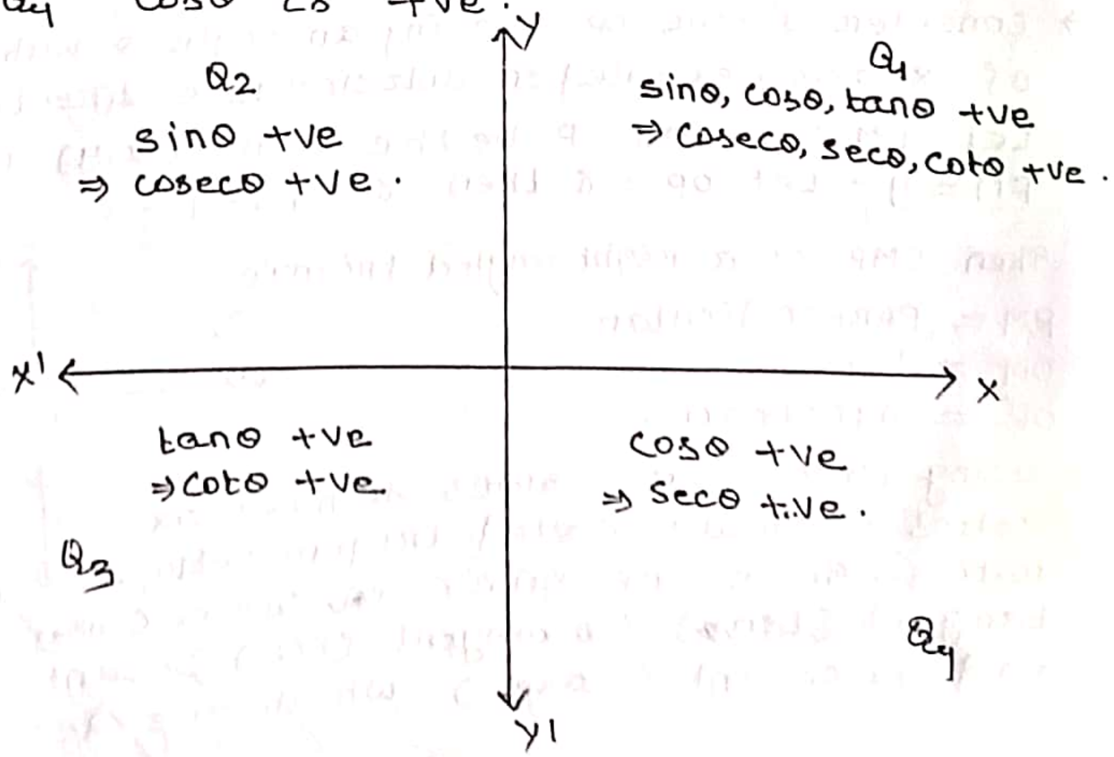
$\operatorname{cosec}\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{perpendicular}}$.

D : ASTC RULE :

$\rightarrow$ A = All
S = sine
T = tangent
C = cosine.

$\rightarrow$ Out of three T-ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ in Q_1, All t-ratios are +ve.

Q_2, $\sin\theta$ is +ve
Q_3 $\tan\theta$ is +ve
Q_4 $\cos\theta$ is +ve.



E : T-ratios of Allied Angles:

→ If sum or difference of two angles is either zero or a multiple of $90°$, then the angles are called Allied angles.

→ $\theta, -\theta, 90-\theta, 90+\theta, 180-\theta, 180+\theta, 270-\theta, 270+\theta, 360-\theta, 360+\theta$ are Allied angles.

E_1 : T-ratios of Angle $(-\theta)$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\text{cosec}(-\theta) = -\text{cosec}\theta$

$\sec(-\theta) = \sec\theta$

$\cot(-\theta) = -\cot\theta$

E_2 : T-ratios of Angle $(90°-\theta)/(\frac{\pi}{2}-\theta)$

$\sin(90-\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$

$\tan(90-\theta) = \cot\theta$

$\cot(90-\theta) = \tan\theta$

$\sec(90-\theta) = \text{cosec}\theta$

$\text{cosec}(90-\theta) = \sec\theta$

E_3 : T-ratios of Angle $(90°+\theta)/(\frac{\pi}{2}+\theta)$

$\sin(90+\theta) = \cos\theta$

$\cos(90+\theta) = -\sin\theta$

$\tan(90+\theta) = -\cot\theta$

$\cot(90+\theta) = -\tan\theta$

$\sec(90+\theta) = -\sec\theta$

$\text{cosec}(90+\theta) = \text{cosec}\theta$

E_4 : T-ratios of angle $(180-\theta)/(\pi-\theta)$

$\sin(180-\theta) = \sin\theta$

$\cos(180-\theta) = -\cos\theta$

$\tan(180-\theta) = -\tan\theta$

$\cot(180-\theta) = -\cot\theta$

$\sec(180-\theta) = -\sec\theta$

$\text{cosec}(180-\theta) = \text{cosec}\theta$.

E_5 : T-ratios of angle $(180+\theta)/\pi+\theta$

$\sin(180+\theta) = -\sin\theta$

$\cos(180+\theta) = -\cos\theta$

$\tan(180+\theta) = \tan\theta$

$\cot(180+\theta) = \cot\theta$

$\sec(180+\theta) = -\sec\theta$

$\text{cosec}(180+\theta) = -\text{cosec}\theta$

E_6 : T-ratios of angle $(270-\theta)/(\frac{3\pi}{2}-\theta)$

$\sin(270-\theta) = -\cos\theta$

$\cos(270-\theta) = -\sin\theta$

$\tan(270-\theta) = \cot\theta$

$\cot(270-\theta) = \tan\theta$

$\sec(270-\theta) = -\text{cosec}\theta$

$\text{cosec}(27-\theta) = -\sec\theta$

E_7 : T-ratios of angle $(270+\theta)/(\frac{3\pi}{2}+\theta)$

$\sin(270+\theta) = -\cos\theta$

$\cos(270+\theta) = \sin\theta$

$\tan(270+\theta) = -\cot\theta$

$\cot(270+\theta) = -\tan\theta$

$\sec(270+\theta) = \text{cosec}\theta$

$\text{cosec}(270+\theta) = -\sec\theta$

E8: T-ratios of Angle $(360-\theta)/(2\pi-\theta)$

$\sin(360^\circ-\theta) = -\sin\theta$ $\quad\quad$ $\cot(360-\theta) = -\cot\theta$

$\cos(360-\theta) = \cos\theta$ $\quad\quad$ $\sec(360-\theta) = \sec\theta$

$\tan(360-\theta) = -\tan\theta$ $\quad\quad$ $\operatorname{cosec}(360-\theta) = -\operatorname{cosec}\theta$

E9: T-ratios of Angle $(360+\theta)/(2\pi+\theta)$

$\sin(360+\theta) = \sin\theta$ $\quad\quad$ $\cot(360+\theta) = \cot\theta$

$\cos(360+\theta) = \cos\theta$ $\quad\quad$ $\sec(360+\theta) = \sec\theta$

$\tan(360+\theta) = \tan\theta$ $\quad\quad$ $\operatorname{cosec}(360+\theta) = \operatorname{cosec}\theta$

NOTE:

i) T-ratios of Angles $2n\pi\pm\theta, n=1,2,3$ ------ remain same for sign of RHS follow ASTC Rule.

sin → sin, cos → cos, tan → tan, cot → cot, sec → sec, cosec → cosec.

ii) T-ratios of angles → $n\pi\pm\theta, n=0,1,2,3$ ------ remain same and for sign on RHS follow ASTC Rule.

(iii) T-ratios of angles $\frac{n\pi}{2}\pm\theta, n=1,3,5$ ---- changes like sin ↔ cos, tan ↔ cot, sec ↔ cosec

For sign on RHS, follow ASTC Rule.

F: T-ratios OF Angles $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ, 135^\circ, 150^\circ, 180^\circ$

→ Note that $\sin 0^\circ = 0$, $\sin 30^\circ = \frac{1}{2}$, $\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\sin 90^\circ = 1$, $\sin 120^\circ = \frac{\sqrt{3}}{2}$, $\sin 135 = \frac{1}{\sqrt{2}}$, $\sin 150^\circ = \frac{1}{2}$

$\sin 180^\circ = 0$

→ To find $\cos\theta$, Apply $90-\theta$ formula for acute angle, 90° and $180-\theta$ for obtuse angle.

→ For othe t-ratios apply $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ and if it is obtuse angle then first apply $180-\theta$ formula.

E10: T-ratios of angles $n\pi\pm\theta$, $\left(\frac{n\pi}{2}+\theta\right)$

→ $\sin(n\pi+\theta) = (-1)^n \sin\theta$

$\cos(n\pi+\theta) = (-1)^n \cos\theta$

$\tan(n\pi+\theta) = \tan\theta$

→ $\sin(n\pi-\theta) = (-1)^{n+1}\sin\theta$

$\cos(n\pi-\theta) = (-1)^{n}\cos\theta$

$\tan(n\pi-\theta) = -\tan\theta$.

→ $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{(n-1)/2}\cos\theta$, n is odd

$= (-1)^{n/2}\sin\theta$, n is even

$\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$, n is odd

$= (-1)^{\frac{n}{2}}\cos\theta$, n is even

$\tan\left(\frac{n\pi}{2}+\theta\right) = -\cot\theta$, n is odd

$= \tan\theta$, n is even.

Ex-3: Express $\cos 1400°$ in terms of an acute angle.

Ans: $\cos 1400° = \cos(6\pi+320)$

$= \cos 320$ ($1400° \in Q_4$ and $(2n\pi+\theta)$ form)

$= \cos\left(\frac{3\pi}{2}+50\right)$ ($320° \in Q_4$ and $\left(\frac{n\pi}{2}+\theta\right.$ form)

$= \sin 50$

Ex-4 Find the value of (a) $\sin 480°$ (b) $\operatorname{cosec}^2\left(\frac{7\pi}{6}\right)$ (c) $\cot\left(\frac{5\pi}{6}\right)$

Soln:

(a) $\sin 480° = \sin(2\pi+120)$

$= \sin 120° = \frac{\sqrt{3}}{2}$. $480° \in Q_2$ and $2n\pi+\theta$ form

(b) $\operatorname{cosec}^2\left(\frac{7\pi}{6}\right) = \operatorname{cosec}^2\left(\pi+\frac{\pi}{6}\right)$

$= \left[-\operatorname{cosec}\frac{\pi}{6}\right]^2$ ($\frac{7\pi}{6} \in Q_3$, $n\pi+\theta$ form)

$= (-2)^2 = 4$ (Ans)

(c) $\cot\left(\frac{5\pi}{6}\right) = \cot\left(\pi-\frac{\pi}{6}\right)$

($\frac{5\pi}{6} \in Q_2$, $n\pi-\theta$ form)

$= -\cot\frac{\pi}{6} = -\sqrt{3}$ (Ans)

Ex-5 Prove that $2\sin^2\frac{3\pi}{4} + 2\cos^2\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{3}\right) = 6$.

LHS $= 2\sin^2\left(\frac{3\pi}{4}\right) + 2\cos^2\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{3}\right)$

$= 2\times\left(\frac{1}{\sqrt{2}}\right)^2 + 2\times\left(\frac{1}{\sqrt{2}}\right)^2 + (2)^2$

$= 1+1+4 = 6 =$ RHS. □

Ex-6 Prove that $\tan 225^\circ \cot 405^\circ + \tan 1485^\circ * \cot 315^\circ = 0$

Soln: $\tan 225^\circ = \tan(180+45) = \tan 45^\circ = 1.$ $\quad (\because 225^\circ \in Q_3, n\pi+\theta \text{ form})$

$\cot(405) = \cot(360+45) = \cot 45 = 1$ $\quad (\because 405^\circ \in Q_1, 2n\pi+\theta \text{ form})$

~~Cot (1485°)~~

$\tan 1485 = \tan(4\times360+45^\circ) = \tan 45 = 1$ $\quad (1485^\circ \in Q_1, 2n\pi+\theta)$

$\cot 315^\circ = \cot(360-45^\circ) = -\cot 45 = -1$ $\quad (315^\circ \in Q_4, 2n\pi-\theta)$

$LHS = \tan 225^\circ \cdot \cot 405^\circ + \tan 1485^\circ \cdot \cot 315$

$= (1)(1) + (1)(-1) = 1-1 = 0 = RHS$ □.

Ex-7: Prove that $\dfrac{\sin(\theta-\frac{\pi}{2})}{\cos(\pi-\theta)} + \dfrac{\tan(\frac{\pi}{2}-\theta)}{\cot(\pi+\theta)} + \dfrac{\operatorname{cosec}(\frac{\pi}{2}+\theta)}{\sec(2\pi-\theta)} = 3$

$$LHS = \frac{\sin(\theta-\frac{\pi}{2})}{\cos(\pi-\theta)} + \frac{\tan(\frac{\pi}{2}-\theta)}{\cot(\pi+\theta)} + \frac{\operatorname{cosec}(\frac{\pi}{2}+\theta)}{\sec(2\pi-\theta)}$$

$$= \frac{-\cos\theta}{-\cos\theta} + \frac{\cot\theta}{\cot\theta} + \frac{\sec\theta}{\sec\theta}$$

$= 1 + 1 + 1 = 3 = RHS$ □.

Ex-8 For the triangle PQR, prove that $\sin(Q+R) = \sin P$.

Soln In the triangle PQR

$m\angle P + m\angle Q + m\angle R = 180^\circ$

$\Rightarrow P + Q + R = 180^\circ$

$\Rightarrow Q + R = 180 - P$

$\Rightarrow \sin(Q+R) = \sin(180-P) = \sin P$ □.

HOME TASK-2

1) Find the value of following T-ratios:

(i) $\sec(-2025^\circ)$ (ii) $\cos(-5\pi+\theta)$ (iii) $\cot\left(\frac{5\pi}{6}\right)$

2. Simplify $\tan\left(\frac{\pi}{20}\right) \cdot \tan\left(\frac{3\pi}{20}\right) \cdot \tan\left(\frac{5\pi}{20}\right) \cdot \tan\left(\frac{7\pi}{20}\right) \cdot \tan\left(\frac{9\pi}{20}\right)$

(3) In a cyclic quadrilateral ABCD, prove that

$\cos(180-A) - \sin(90+B) + \cos(180+C) - \sin(90+D) = 0$

(4) prove that $\dfrac{\tan(\frac{\pi}{2}+\theta)}{\cot(\pi-\theta)} + \dfrac{\sin(\pi+\theta)}{\sin(2\pi-\theta)} + \dfrac{\cos(2\pi+\theta)}{\sin(\frac{\pi}{2}+\theta)} = 3$

(5) prove that $\tan 660^\circ \cdot \cot 1320^\circ + \cot 390^\circ \cdot \tan 210^\circ = 0$

C_1 Identities Relation:

i) $\sin^2\theta + \cos^2\theta = 1$

ii) $\sec^2\theta - \tan^2\theta = 1$

iii) $\mathrm{cosec}^2\theta - \cot^2\theta = 1$

C_2 : Reciprocal Relation:

i) $\sin\theta = \frac{1}{\mathrm{cosec}\theta}$ or $\mathrm{cosec}\theta = \frac{1}{\sin\theta}$ or $\sin\theta.\mathrm{cosec}\theta = 1$

ii) $\cos\theta = \frac{1}{\sec\theta}$ or $\sec\theta = \frac{1}{\cos\theta}$ or $\cos\theta.\sec\theta = 1$

iii) $\tan\theta = \frac{1}{\cot\theta}$ or $\cot\theta = \frac{1}{\tan\theta}$ or $\tan\theta.\cot\theta = 1$

C_3: Division Relation

i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

C_4: SUM AND DIFFERENCE FORMULAE:

i) $\sin(A+B) = \sin A.\cos B + \cos A.\sin B$

ii) $\sin(A-B) = \sin A.\cos B - \cos A.\sin B$

iii) $\cos(A+B) = \cos A.\cos B - \sin A.\sin B$

iv) $\cos(A-B) = \cos A.\cos B + \sin A.\sin B$

v) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A.\tan B}$

vi) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A.\tan B}$

vii) $\cot(A+B) = \frac{\cot A.\cot B - 1}{\cot B + \cot A}$

viii) $\cot(A-B) = \frac{\cot A.\cot B + 1}{\cot B - \cot A}$

ix) $\sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$.

x) $\cos(A+B).\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.

Ex-9 : Evaluate $\sin 75°$

Soln: $\sin 75° = \sin(45° + 30°) = \sin 45°.\cos 30° + \cos 45°.\sin 30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+1)}{2\sqrt{2}.\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ (ANS)

Ex-10: Prove that $\tan 57° = \frac{\cos 12° + \sin 12°}{\cos 12° - \sin 12°}$

$LHS = \tan 57° = \tan(45° + 12°) = \frac{\tan 45 + \tan 12}{1 - \tan 45.\tan 12} = \frac{1 + \tan 12}{1 - \tan 12}$

$= \frac{1 + \frac{\sin 12}{\cos 12}}{1 - \frac{\sin 12}{\cos 12}} = \frac{\cos 12 + \sin 12}{\cos 12} \Big/ \frac{\cos 12 - \sin 12}{\cos 12} = \frac{\cos 12 + \sin 12}{\cos 12} \times \frac{\cos 12}{\cos 12 - \sin 12}$

$= \frac{\cos 12 + \sin 12}{\cos 12 - \sin 12} = RHS$ □.

Ex. 11 Prove that $\tan 50^\circ = \tan 40^\circ + 2\tan 10^\circ$

Soln: $50^\circ = 40^\circ + 10^\circ$

$\Rightarrow \tan 50^\circ = \tan(40^\circ + 10^\circ)$

$\Rightarrow \tan 50^\circ = \dfrac{\tan 40^\circ + \tan 10^\circ}{1 - \tan 40 . \tan 10} \Rightarrow \tan 50^\circ (1 - \tan 40^\circ . \tan 10^\circ) = \tan 40 + \tan 10^\circ$

$\Rightarrow \tan 50^\circ - \tan 50^\circ . \tan 40^\circ . \tan 10^\circ = \tan 40^\circ + \tan 10^\circ$

$\Rightarrow \tan 50^\circ - \tan(90^\circ - 40^\circ) . \tan 40^\circ . \tan 10^\circ = \tan 40^\circ + \tan 10^\circ$

$\Rightarrow \tan 50^\circ - \cot 40^\circ . \tan 40^\circ . \tan 10^\circ = \tan 40^\circ + \tan 10^\circ$

$\Rightarrow \tan 50^\circ - \tan 10^\circ = \tan 40^\circ + \tan 10^\circ$ $(\because \tan 40^\circ . \cot 40^\circ = 1)$

$\Rightarrow \tan 50^\circ = \tan 40^\circ + \tan 10^\circ + \tan 10^\circ$

$\Rightarrow \tan 50^\circ = \tan 40^\circ + 2\tan 10^\circ$ $\square$

Ex-12 Evaluate $\cos^2\left(\frac{\pi}{4} + x\right) - \sin^2\left(\frac{\pi}{4} - x\right)$

Soln: $\cos^2\left(\frac{\pi}{4} + x\right) - \sin^2\left(\frac{\pi}{4} - x\right)$

$= \cos^2 A - \sin^2 B$ (put $\frac{\pi}{4} + x = A$, $\frac{\pi}{4} - x = B$)

$= \cos(A+B) . \cos(A-B)$

$= \cos \frac{2\pi}{4} . \cos 2x$ $\because A+B = \frac{2\pi}{4}$, $A-B = 2x$

$= \cos \frac{\pi}{2} . \cos 2x = 0 \times \cos 2x = 0$ (Ans).

Ex-13: If $A+B = 45^\circ$ then show that $(1+\tan A)(1+\tan B) = 2$.

Soln: $A+B = 45^\circ \Rightarrow \tan(A+B) = \tan 45^\circ \Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A . \tan B} = 1$

$\Rightarrow \tan A + \tan B = 1 - \tan A . \tan B \Rightarrow \tan A + \tan B + \tan A . \tan B = 1$

$\Rightarrow 1 + \tan A + \tan B + \tan A . \tan B = 1+1$

$\Rightarrow 1(1+\tan A) + \tan B(1+\tan A) = 2 \Rightarrow (1+\tan A)(1+\tan B) = 2$ $\square$

Ex-14: In any triangle ABC if $\cos A = \cos B . \cos C$ then show that $\tan B + \tan C = \tan A$.

Soln: In any triangle ABC, $A+B+C = \pi$ ——(1)

Again $\cos A = \cos B . \cos C$ ——(2)

From (1) $A+B+C = \pi \Rightarrow B+C = \pi - A$

$\Rightarrow \sin(B+C) = \sin(\pi - A)$

$\Rightarrow \sin B . \cos C + \cos B . \sin C = \sin A$

$\Rightarrow \dfrac{\sin B . \cos C + \cos B . \sin C}{\cos A} = \dfrac{\sin A}{\cos A}$

$$\Rightarrow \frac{\sin B.\cos C}{\cos A} + \frac{\cos B.\sin C}{\cos A} = \tan A$$

$$\Rightarrow \frac{\sin B.\cos C}{\cos B.\cos C} + \frac{\cos B.\sin C}{\cos B.\cos C} = \tan A$$

$$\Rightarrow \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \tan A \Rightarrow \tan B + \tan C = \tan A \quad \square$$

HOME TASK-3

1) Find the value of (a) $\cos 15^\circ$ (b) $\sin 15^\circ$ (c) $\tan 75^\circ$ (d) $\cos 75^\circ$ (e) $\cot 15^\circ$

2) Show that
i) $\tan 15^\circ = 2-\sqrt{3}$ (ii) $\tan 75^\circ + \cot 75^\circ = 4$

3) Show that (i) $\frac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \tan 54^\circ$ (ii) $\frac{\cos 18^\circ + \sin 18^\circ}{\cos 18^\circ - \sin 18^\circ} = \tan 63^\circ$

4. Find (i) $\frac{\tan 20^\circ + \tan 25^\circ}{1 - \tan 20^\circ . \tan 25^\circ}$ (ii) $\sin 35^\circ . \cos 25^\circ + \cos 35^\circ . \sin 25^\circ$

5. If $\tan A = \frac{5}{6}$, $\tan B = \frac{1}{11}$ then prove that $A+B = \frac{\pi}{4}$

6. Show that $\frac{\sin(A-B)}{\cos A.\cos B} + \frac{\sin(B-C)}{\cos B.\cos C} + \frac{\sin(C-A)}{\cos C.\cos A} = 0$

7. (i) If $A+B = 45^\circ$ then show that $(\cot A - 1)(\cot B - 1) = 2$.
(ii) Show that $(1+\tan 25)(1+\tan 20) = 2$
(iii) Show that $(\cot 23^\circ - 1)(\cot 22^\circ - 1) = 2$

8. If $\tan A + \tan B = p$, $\cot A + \cot B = q$ then show that $\cot(A+B) = \frac{1}{p} - \frac{1}{q}$

9. If $A+B+C = \pi$ & $\cos A = \cos B.\cos C$ then show that $\tan B.\tan C = 2$

10. Show that $\tan 3\theta - \tan\theta - \tan 2\theta = \tan\theta.\tan 2\theta.\tan 3\theta$

11. In any ΔABC, show that $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$.

H: TRANSFORMATION OF SUM INTO PRODUCT AND VICE-VERSA.

i) $\sin(A+B) + \sin(A-B) = 2\sin A.\cos B$

ii) $\sin(A+B) - \sin(A-B) = 2\cos A.\sin B$

iii) $\cos(A+B) + \cos(A-B) = 2\cos A.\cos B$

(iv) $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$

(v) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$

(vi) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$

vii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

viii) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$

I: T-RATIOS OF MULTIPLE ANGLE (2A, 3A):

i) $\sin 2A = 2\sin A \cdot \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

ii) $\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

iv) $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

v) $\sin 3A = 3\sin A - 4\sin^3 A$

vi) $\cos 3A = 4\cos^3 A - 3\cos A$

vii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

viii) $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

(ix) $4\sin A \cdot \sin(60-A) \cdot \sin(60+A) = \sin 3$

(x) $4\cos A \cdot \cos(60+A) \cdot \cos(60-A) = \cos 3$

J: T-RATIOS OF SUB MULTIPLE ANGLES

J_1: ANGLE A in A/2

i) $\sin A = 2\sin\left(\frac{A}{2}\right) \cdot \cos\left(\frac{A}{2}\right)$

(ii) $\cos A = \cos^2(A/2) - \sin^2(A/2) = 2\cos^2(A/2) - 1 = 1 - 2\sin^2(A/2)$

$= \dfrac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2(A/2)}$

(iii) $\tan A = \dfrac{2\tan(A/2)}{1-\tan^2(A/2)}$ (iv) $\cot A = \dfrac{\cot^2(A/2) - 1}{2\cot(A/2)}$

J_2: ANGLE A INTERMS OF A/3:

i) $\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$

(ii) $\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$

iii) $\tan A = \dfrac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1 - 3\tan^2\left(\frac{A}{3}\right)}$

(iv) $\cot A = \dfrac{\cot^3(A/3) - 3\cot(A/3)}{3\cot^2(A/3) - 1}$

J_3: ANGLE A in terms of 2A

i) $2\cos^2 A = 1 + \cos 2A$

ii) $2\sin^2 A = 1 - \sin 2A$

J_4: ANGLE A/2 in terms of A.

i) $2\cos^2(A/2) = 1 + \cos A$

ii) $2\sin^2(A/2) = 1 - \sin A$

iii) $\tan\left(\frac{A}{2}\right) = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$

iv) $\cot\left(\frac{A}{2}\right) = \frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A}$.

K: MAXIMUM AND MINIMUM VALUE OF $a\sin\theta + b\cos\theta$

i) max. $\sin\theta = 1$ for $\theta = \frac{\pi}{2}$
min $\sin\theta = -1$ for $\theta = \frac{3\pi}{2}$

ii) max. $\cos\theta = 1$ for $\theta = 0^\circ$
min $\cos\theta = -1$ for $\theta = \pi$

iii) To find maximum & minimum value of $P(\theta) = a\sin\theta + b\cos\theta$,
Let $a = r\cos\alpha$, $b = r\sin\alpha$ then $r = \sqrt{a^2+b^2}$ and

$P(\theta) = r\cos\alpha.\sin\theta + r\sin\alpha.\cos\theta$

$= r(\cos\alpha.\sin\theta + \sin\alpha.\cos\theta) = r\sin(\theta+\alpha)$ —— ①

$\because$ max. $\sin(\theta+\alpha) = 1 \Rightarrow$ max. $P(\theta) = r.1 = r = \sqrt{a^2+b^2}$

& min $\sin(\theta+\alpha) = -1 \Rightarrow$ min. $P(\theta) = r.(-1) = -r = -\sqrt{a^2+b^2}$.

EX-15: Prove that $\sin(A+B).\sin(A-B) = \sin^2A - \sin^2B = \cos^2B - \cos^2A$.

Soln $\sin(A+B).\sin(A-B)$ ---------(i)

$= (\sin A.\cos B + \cos A.\sin B).(\sin A.\cos B - \cos A.\sin B)$

$= (\sin A.\cos B)^2 - (\cos A.\sin B)^2$

$= \sin^2A.\cos^2B - \cos^2A.\sin^2B = \sin^2A(1-\sin^2B) - \sin^2B(1-\sin^2A)$

$= \sin^2A - \sin^2A.\sin^2B - \sin^2B + \sin^2A.\sin^2B$

$= \sin^2A - \sin^2B$ ---------- (ii)

$= (1-\cos^2A) - (1-\cos^2B) = 1-\cos^2A - 1 + \cos^2B$

$= \cos^2B - \cos^2A$ ---------- (iii)

From (i), (ii) and (iii) we have

$\sin(A+B).\sin(A-B) = \sin^2A - \sin^2B = \cos^2B - \cos^2A$ □

Ex-16: Prove that $4\sin A.\sin(60-A).\sin(60+A) = \sin 3A$.

LHS $= 4\sin A.\sin(60-A).\sin(60+A)$

$= 4\sin A.(\sin^2 60 - \sin^2 A)$

$= 4\sin A.\left\{\left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2A\right\} = 4\sin A\left\{\frac{3}{4} - \sin^2A\right\}$

$= 4\sin A.\left\{\frac{3-4\sin^2A}{4}\right\} = 3\sin A - 4\sin^3A = \sin 3A =$ RHS □

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Ex-17: Show that the equation $\cos\theta = a + \frac{1}{a}$ doesnot have a Solution if $a \neq 0$ is real.

Soln: $a \in R$ and $a \neq 0$

$$\left(a+\frac{1}{a}\right)^2 = \left(a-\frac{1}{a}\right)^2 + 4 \cdot a \cdot \frac{1}{a} = \left(a-\frac{1}{a}\right)^2 + 4 \geq 0 + 4 = 4$$

$$\because \left(a-\frac{1}{a}\right)^2 \geq 0$$

$\Rightarrow \left(a+\frac{1}{a}\right)^2 \geq 4 \Rightarrow \cos^2\theta \geq 4 \Rightarrow \cos^2\theta - 4 \geq 0 \Rightarrow (\cos\theta+2)(\cos\theta-2) \geq 0$

$\Rightarrow \cos\theta + 2 \geq 0$ and $\cos\theta - 2 \geq 0$ or $\cos\theta + 2 \leq 0$ and $\cos\theta - 2 \leq 0$

$\Rightarrow \cos\theta \geq -2$ and $\cos\theta \geq 2$ or $\cos\theta \leq -2$ and $\cos\theta \leq 2$

$\Rightarrow \cos\theta \geq 2$ or $\cos\theta \leq -2$ which is impossible for $-1 \leq \cos\theta \leq 1$

Hence the result.

Ex-18: If $A = \cos^2\theta + \sin^4\theta$ then prove that for all values of θ, $\frac{3}{4} \leq A \leq 1$

Soln: $A = \cos^2\theta + \sin^4\theta \Rightarrow A = 1 - \sin^2\theta + \sin^4\theta$

$\Rightarrow \sin^4\theta - \sin^2\theta + (1-A) = 0$

$$\Rightarrow \sin^2\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (1-A)}}{2 \times 1} = \frac{1 \pm \sqrt{1-4+4A}}{2}$$

$$\Rightarrow \sin^2\theta = \frac{1 \pm \sqrt{4A-3}}{2} \quad \text{——(1)}$$

since $\sin^2\theta$ is Real $\Rightarrow$ Discriminant $\geq 0 \Rightarrow 4A - 3 \geq 0$

$\Rightarrow 4A \geq 3 \Rightarrow A \geq \frac{3}{4} \Rightarrow \frac{3}{4} \leq A$ ——(2)

Again $-1 \leq \sin\theta \leq 1 \Rightarrow \sin^2\theta \leq 1 \Rightarrow \frac{1 \pm \sqrt{4A-3}}{2} \leq 1$

$\Rightarrow 1 \pm \sqrt{4A-3} \leq 2 \Rightarrow \pm\sqrt{4A-3} \leq 1 \Rightarrow 4A - 3 \leq 1 \Rightarrow 4A \leq 4$

$\Rightarrow A \leq 1$ ——(3)

combining (2) & (3) we have $\frac{3}{4} \leq A \leq 1$ $\square$

Ex-19: Show that $2^n \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdots\cdots \cos 2^{n-1}\theta = 1$ If $\theta = \frac{\pi}{2^n+1}$

Soln $\theta = \frac{\pi}{2^n+1} \Rightarrow 2^n\theta + \theta = \pi \Rightarrow 2^n\theta = (\pi - \theta)$ ——(1)

LHS $= 2^n \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdots\cdots\cdots \cos 2^{n-1}\theta$

$$= \frac{2^{n-1}(2\sin\theta \cdot \cos\theta) \cdot \cos 2\theta \cdot \cos 2^2\theta \cdots\cdots \cos 2^{n-1}\theta}{\sin\theta}$$

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$$= \frac{2^{n-1} \sin 2\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdots\cdots \cos 2^{n-1}\theta}{\sin\theta}$$

$$= \frac{2^{n-2}\; 2\sin 2\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdots\cdots \cos 2^{n-1}\theta}{\sin\theta}$$

$$= \frac{2^{n-2} \sin 2^2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdots\cdots \cos 2^{n-1}\theta}{\sin\theta}$$

$$= \frac{2^{n-(n-1)} \cdot \sin 2^{n-1}\theta \cdot \cos 2^{n-1}\theta}{\sin\theta}$$ (proceeding in this way n-1 times)

$$= \frac{2\cdot \sin 2^{n-1}\theta \cdot \cos 2^{n-1}\theta}{\sin\theta} = \frac{\sin(2\times 2^{n-1}\theta)}{\sin\theta} = \frac{\sin(2^n\theta)}{\sin\theta}$$

$$= \frac{\sin(\pi-\theta)}{\sin\theta}$$ (using eq^n(1)) $= \frac{\sin\theta}{\sin\theta} = 1 = RHS$ □

Ex-20 prove that $\sqrt{\frac{1+\sin A}{1-\sin A}} = \tan\left(\frac{\pi}{4}+\frac{A}{2}\right)$.

Sol^n: LHS $= \sqrt{\frac{1+\sin A}{1-\sin A}} = \sqrt{\frac{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}-2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}} = \sqrt{\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2}{\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2}}$

$$= \frac{\cos\left(\frac{A}{2}\right)+\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)-\sin\left(\frac{A}{2}\right)} = \frac{1+\tan\left(\frac{A}{2}\right)}{1-\tan\left(\frac{A}{2}\right)}$$ Dividing N^r & D^r by $\cos\left(\frac{A}{2}\right)$

$$= \frac{\tan\frac{\pi}{4}+\tan\frac{A}{2}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{A}{2}} = \tan\left(\frac{\pi}{4}+\frac{A}{2}\right) = RHS$$ $\because \tan\frac{\pi}{4}=1$ □

Ex-21: Show that $\tan\left(\frac{A}{2}\right) = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$

Sol^n: $\tan\left(\frac{A}{2}\right)$ ---------(i)

$$= \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} = \frac{2\sin^2(A/2)}{2\sin(A/2)\cdot\cos(A/2)} = \frac{1-\cos\left(2\times\frac{A}{2}\right)}{\sin\left(2\times\frac{A}{2}\right)} = \frac{1-\cos A}{\sin A}$$ --- (ii)

$$= \frac{(1-\cos A)\times(1+\cos A)}{\sin A(1+\cos A)} = \frac{1-\cos^2 A}{\sin A(1+\cos A)} = \frac{\sin^2 A}{\sin A(1+\cos A)} = \frac{\sin A}{1+\cos A}$$ ——(iii)

From eq^n (i),(ii) & (iii) we have $\tan\left(\frac{A}{2}\right) = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$ □

Ex-22 Find maximum and minimum value of $8\cos x-15\sin x-12$.

Sol^n: Let $p(x) = 8\cos x - 15\sin x - 12$

put $8 = r\sin\alpha$, $15 = r\cos\alpha$ then $r = \sqrt{8^2+15^2} = \sqrt{289} = 17$.

Now $p(x) = r\sin\alpha\cdot\cos x - r\cos\alpha\cdot\sin x - 12$

$= r(\sin\alpha\cdot\cos x - \cos\alpha\cdot\sin x) - 12 = r\sin(\alpha-x)-12$

$= 17\sin(\alpha-x)-12$ ———(1)

$\because$ max. $\sin(\alpha-x) = 1 \Rightarrow$ max. $p(x) = 17\times1-12 = 5$

min $\sin(\alpha-x) = -1 \Rightarrow$ min $p(x) = 17\times(-1)-12 = -29$.

$\therefore$ maximum value $= 5$, minimum value $= -29$ (Ans).

EX-23: Find maximum & minimum value of $\sin\theta.\cos\theta$ and hence find for what values of θ it is maximum and minimum.

Soln: Let $P(\theta) = \sin\theta.\cos\theta = \frac{1}{2}(2\sin\theta.\cos\theta) = \frac{1}{2}\sin 2\theta$ ——(i)

$\because$ $\max.\sin 2\theta = 1 \Rightarrow \max.P(\theta) = \frac{1}{2}\times 1 = \frac{1}{2}$ and it is maximum for $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

Again $\min \sin 2\theta = -1 \Rightarrow \min P(\theta) = \frac{1}{2}\times(-1) = \frac{-1}{2}$ and it is minimum for $2\theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{4}$.

$\therefore$ maximum value $= \frac{1}{2}$ for $\theta = \frac{\pi}{4}$ and minimum value $= -\frac{1}{2}$ for $\theta = \frac{3\pi}{4}$ (Ans).

EX-24: Show that $\cos 20^\circ.\cos 40^\circ.\cos 60^\circ.\cos 80^\circ = \frac{1}{16}$.

LHS $= \cos 20^\circ.\cos 40^\circ.\cos 60^\circ.\cos 80^\circ$

$= \frac{1}{2}\cos 20^\circ.\cos 40^\circ.\cos 80^\circ$ $\quad \because \cos 60^\circ = \frac{1}{2}$

$= \frac{1}{2}\cos 20^\circ.\cos(60-20).\cos(60+20)$

$= \frac{1}{2}\cos A.\cos(60-A).\cos(60+A)$ (put $A = 20^\circ$)

$= \frac{1}{2}\cos A.(\cos^2 A - \sin^2 60) = \frac{1}{2}\cos A.\left(\cos^2 A - \left(\frac{\sqrt{3}}{2}\right)^2\right)$

$= \frac{1}{2}\cos A\left\{\cos^2 A - \frac{3}{4}\right\} = \frac{1}{2}\cos A\left(\frac{4\cos^2 A - 3}{4}\right)$

$= \frac{1}{8}(4\cos^3 A - 3\cos A) = \frac{1}{8}\cos 3A = \frac{1}{8}\cos(3\times 20) = \frac{1}{8}\cos 60^\circ$

$= \frac{1}{8}\times\frac{1}{2} = \frac{1}{16}$ = RHS $\square$

EX-25: If $\sin A = K\sin B$ then prove that $\tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\tan\frac{1}{2}(A+B)$

Soln: $\sin A = K\sin B \Rightarrow \frac{\sin A}{\sin B} = K \Rightarrow \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{K+1}{K-1}$ (by Componendo and Dividendo)

$\Rightarrow \frac{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)} = \frac{K+1}{K-1} \Rightarrow \tan\left(\frac{A+B}{2}\right).\cot\left(\frac{A-B}{2}\right) = \frac{K+1}{K-1}$

$\Rightarrow \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} = \frac{K+1}{K-1} \Rightarrow (K+1)\tan\left(\frac{A-B}{2}\right) = (K-1)\tan\left(\frac{A+B}{2}\right)$

$\Rightarrow \tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\tan\frac{1}{2}(A+B)$ $\square$

Ex-26: If $A+B+C=\pi$ then prove that $\sin 2A+\sin 2B-\sin 2C = 4\cos A\cos B.\sin C$

Soln: $A+B+C=\pi$ —— (1)

LHS $= \sin 2A+\sin 2B-\sin 2C = 2\sin\left(\frac{2A+2B}{2}\right).\cos\left(\frac{2A-2B}{2}\right)-\sin 2C$

$= 2\sin(A+B).\cos(A-B)-\sin 2C = 2\sin(\pi-C).\cos(A-B)-\sin 2C$ using (1)

$= 2\sin C.\cos(A-B)-2\sin C.\cos C = 2\sin C\{\cos(A-B)-\cos C\}$

$= 2\sin C\{\cos(A-B)-\cos[\pi-(A+B)]\}$ using (1)

$= 2\sin C\{\cos(A-B)+\cos(A+B)\} = 2\sin C.2\cos A\cos B = 4\cos A.\cos B.\sin C$

$=$ RHS □

Ex-27: Show that $\cot 7\frac{1}{2}^\circ = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$.

Soln: $\cot 7\frac{1}{2}^\circ = \cot\left(\frac{15}{2}\right)^\circ = \frac{1+\cos 15}{\sin 15}$ $\quad\because \cot\frac{A}{2} = \frac{1+\cos A}{\sin A}$

$= \frac{1+\cos(45-30)}{\sin(45-30)} = \frac{1+(\cos 45.\cos 30+\sin 45^\circ.\sin 30^\circ)}{\sin 45.\cos 30-\cos 45.\sin 30^\circ}$

$= \frac{1+\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}.\frac{1}{2}}{\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}.\frac{1}{2}} = \left(\frac{2\sqrt{2}+\sqrt{3}+1}{2\sqrt{2}}\right)\Big/\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

$= \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} = \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{(\sqrt{3})^2-(1)^2}$

$= \frac{2\sqrt{6}+2\sqrt{3}+2\sqrt{2}+4}{3-1} = \frac{2(\sqrt{6}+\sqrt{3}+\sqrt{2}+2)}{2} = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$ □

Ex-28: Show that $2\cos\left(\frac{\pi}{16}\right) = \sqrt{2+\sqrt{2+\sqrt{2}}}$

LHS $= 2\cos\left(\frac{\pi}{16}\right) = \sqrt{4\cos^2\left(\frac{\pi}{16}\right)} = \sqrt{2\left(2\cos^2\left(\frac{\pi}{16}\right)\right)} = \sqrt{2\left(1+\cos\left(2\times\frac{\pi}{16}\right)\right)}$

$= \sqrt{2+2\cos\frac{\pi}{8}} = \sqrt{2+\sqrt{4\cos^2\frac{\pi}{8}}} = \sqrt{2+\sqrt{2.\,2\cos^2\frac{\pi}{8}}}$

$= \sqrt{2+\sqrt{2\left(1+\cos\left(2\times\frac{\pi}{8}\right)\right)}} = \sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}} = \sqrt{2+\sqrt{2+2\times\frac{1}{\sqrt{2}}}}$

$= \sqrt{2+\sqrt{2+\sqrt{2}}}$ = RHS □.

Ex-29: Prove that $\sin 65^\circ+\cos 65^\circ = \sqrt{2}\cos 20^\circ$

LHS $= \sin 65^\circ+\cos 65^\circ = \sin(90^\circ-25)+\cos 65^\circ = \cos 25^\circ+\cos 65^\circ$

$= 2\cos\left(\frac{65+25}{2}\right).\cos\left(\frac{65-25}{2}\right) = 2\cos 45^\circ.\cos 20^\circ = 2\times\frac{1}{\sqrt{2}}.\cos 20^\circ = \sqrt{2}\cos 20^\circ$ = RHS

Ex-30: Prove that $\cot 2\theta+\tan\theta = \operatorname{cosec} 2\theta$

LHS $= \cot 2\theta+\tan\theta = \frac{\cos 2\theta}{\sin 2\theta}+\frac{\sin\theta}{\cos\theta} = \frac{\cos 2\theta.\cos\theta+\sin 2\theta.\sin\theta}{\sin 2\theta.\cos\theta} = \frac{\cos(2\theta-\theta)}{\sin 2\theta.\cos\theta}$

$= \frac{\cos\theta}{\sin 2\theta.\cos\theta} = \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta$ = RHS □

Ex-31: Evaluate $\sin\frac{7\pi}{4}\cdot\cos\frac{\pi}{4} - \cos\frac{7\pi}{4}\cdot\sin\frac{\pi}{4}$

Soln: $\sin\frac{7\pi}{4}\cdot\cos\frac{\pi}{4} - \cos\frac{7\pi}{4}\cdot\sin\frac{\pi}{4} = \sin\left(\frac{7\pi}{4}-\frac{\pi}{4}\right) = \sin\left(\frac{6\pi}{4}\right) = \sin\left(\frac{3\pi}{2}\right)$

$= -1$ (Ans)

Ex-32: If $3\cot A\cdot\cot B = 1$ then prove that $\cos(A-B)+2\cos(A+B)=0$

Soln:

$3\cot A\cdot\cot B = 1$

$\Rightarrow \cot A\cdot\cot B = \frac{1}{3} \Rightarrow \frac{\cos A}{\sin A}\cdot\frac{\cos B}{\sin B} = \frac{1}{3} \Rightarrow \frac{\cos A\cdot\cos B}{\sin A\cdot\sin B} = \frac{1}{3}$

$\Rightarrow \frac{\cos A\cdot\cos B + \sin A\cdot\sin B}{\cos A\cdot\cos B - \sin A\cdot\sin B} = \frac{1+3}{1-3}$ (by componendo and dividendo)

$\Rightarrow \frac{\cos(A-B)}{\cos(A+B)} = -2 \Rightarrow \cos(A-B) = -2\cos(A+B) \Rightarrow \cos(A-B)+2\cos(A+B)=0.$

Ex:33: prove that $2\cos\frac{5\pi}{12}\cdot\cos\frac{\pi}{12} = \frac{1}{2}$.

LHS $= 2\cos\frac{5\pi}{12}\cdot\cos\frac{\pi}{12} = \cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)+\cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)$

$= \cos\frac{6\pi}{12} + \cos\frac{4\pi}{12} = \cos\frac{\pi}{2}+\cos\frac{\pi}{3} = 0+\frac{1}{2} = \frac{1}{2} =$ R.H.S.

Ex-34: prove that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

Soln: LHS $= \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cdot\cos\theta}{1+2\cos^2\theta-1} = \frac{2\sin\theta\cdot\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta}$

$= \tan\theta =$ RHS $\square$

Ex-35: prove that $\frac{1+\sin 2\theta}{1-\sin 2\theta} = \tan^2\left(\frac{\pi}{4}+\theta\right)$

LHS $= \frac{1+\sin 2\theta}{1-\sin 2\theta} = \left(1+\frac{2\tan\theta}{1+\tan^2\theta}\right)\Big/\left(1-\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \left(\frac{1+\tan^2\theta+2\tan\theta}{1+\tan^2\theta}\right)\Big/\left(\frac{1+\tan^2\theta-2\tan\theta}{1+\tan^2\theta}\right)$

$= \frac{(1+\tan\theta)^2}{1+\tan^2\theta}\Big/\frac{(1-\tan\theta)^2}{(1+\tan^2\theta)} = \frac{(1+\tan\theta)^2}{(1-\tan\theta)^2} = \left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2$

$= \left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\cdot\tan\theta}\right]^2 = \left[\tan\left(\frac{\pi}{4}+\theta\right)\right]^2 = \tan^2\left(\frac{\pi}{4}+\theta\right) =$ RHS.

Ex-36: prove that $\frac{1+\sin 2A-\cos 2A}{1+\sin 2A+\cos 2A} = \tan A$

LHS $= \frac{1+\sin 2A-\cos 2A}{1+\sin 2A+\cos 2A} = \frac{\sin 2A+(1-\cos 2A)}{\sin 2A+(1+\cos 2A)} = \frac{2\sin A\cdot\cos A+2\sin^2 A}{2\sin A\cdot\cos A+2\cos^2 A}$

$= \frac{2\sin A(\cos A+\sin A)}{2\cos A(\sin A+\cos A)} = \frac{\sin A}{\cos A} = \tan A =$ RHS $\square$

Ex-37: Prove that $\frac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$.

LHS $= \frac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \frac{2\cos\left(\frac{3\theta+\theta}{2}\right).\sin\left(\frac{3\theta-\theta}{2}\right)}{2\cos\left(\frac{3\theta+\theta}{2}\right).\cos\left(\frac{3\theta-\theta}{2}\right)} = \frac{2\cos 2\theta.\sin\theta}{2\cos 2\theta.\cos\theta}$

$= \frac{\sin\theta}{\cos\theta} = \tan\theta =$ RHS.

Ex-38: Prove that $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$

LHS $= \cos 6A = \cos(2\times 3A) = 2\cos^2 3A - 1 = 2\{4\cos^3 A - 3\cos A\}^2 - 1$

$= 2\{16\cos^6 A + 9\cos^2 A - 24\cos^4 A\} - 1$

$= 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1 =$ RHS.

Ex39: Evaluate $\tan\left(\frac{25\pi}{4}\right)$.

Soln $= \tan\left(\frac{25\pi}{4}\right) = \tan\left(6\pi + \frac{\pi}{4}\right)$

$= \tan\frac{\pi}{4} = 1$ $\quad$ ($6\pi + \frac{\pi}{4} \in Q_1$ & $n\pi + \theta$ form)

Ex-40 Evaluate $\tan\left(\frac{\pi}{8}\right)$

Soln: $\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{\pi}{8}\times\frac{180}{\pi}\right)^\circ = \tan\left(\frac{45}{2}\right) = \frac{1-\cos 45^\circ}{\sin 45^\circ} = \frac{1-\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)}$

$= \frac{\sqrt{2}-1}{\sqrt{2}}\times\frac{\sqrt{2}}{1} = \sqrt{2}-1$

HOME TASK-4

No1. Prove that $\cos\theta = \sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right) + \sin\theta$

2. Prove that $\tan 56^\circ = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$

3. In Any triangle ABC prove that $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

4) Prove that (i) $\frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \frac{1}{\cot A}$ (ii) $\frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

5. Prove that $\sin 10^\circ.\sin 30^\circ.\sin 50^\circ.\sin 70^\circ = \frac{1}{16}$

6. Prove that $\tan A + \cot A = 2\operatorname{cosec} 2A$ and hence deduce that $\tan 15^\circ + \cot 15^\circ = 4$

7. Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$

8. Prove that $\sqrt{\frac{1-\cos 2\theta}{2}} = \pm\sin\theta$

9. Prove that $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}$.

10. prove that $\dfrac{\sin 8\theta \cdot \cos\theta - \cos 6\theta \cdot \sin 6\theta}{\cos 2\theta \cdot \cos\theta - \sin 3\theta \cdot \sin 4\theta} = \tan 2\theta$

11. If $(1-e)\tan^2\left(\frac{\beta}{2}\right) = (1+e)\tan^2\left(\frac{\alpha}{2}\right)$ then show that $\cos\beta = \dfrac{\cos\alpha - e}{1 - e\cos\alpha}$

12. show that $\cos^2 A + \cos^2 B + 2\cos A \cdot \cos B \cdot \cos C = \sin^2 C$ If $A+B+C=\pi$

(13) If $A+B+C=\pi$ then prove following:

i) $\cos\left(\frac{A}{2}\right) - \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right) = 4\cos\left(\frac{\pi+A}{4}\right) \cdot \cos\left(\frac{\pi-B}{4}\right) \cdot \cos\left(\frac{\pi+C}{4}\right)$

(ii) $\sin A + \sin B - \sin C = 4\sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \cos\left(\frac{C}{2}\right)$

(iii) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \cdot \sin B \cdot \sin C$

(iv) $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) = 4\sin\left(\frac{\pi-A}{4}\right) \cdot \sin\left(\frac{\pi-B}{4}\right) \cdot \sin\left(\frac{\pi-C}{4}\right) + 1$

14. show that $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

15. Prove that $4\cos A \cdot \cos(60+A) \cdot \cos(60-A) = \cos 3A$

16. Show that $\cot\left(\frac{A}{2}\right) = \dfrac{1+\cos A}{\sin A} = \dfrac{\sin A}{1-\cos A}$.

17. Show that $\tan\left(\frac{A}{2}\right) = \sqrt{\dfrac{1-\cos A}{1+\cos A}}$

18: Find maximum and minimum value of

(i) $5\sin x + 12\cos x$ (ii) $3\sin x + 4\cos x - 4$

(19) prove that

i) $\cos 4A - \cos 4B = 8(\cos A - \cos B)(\cos A + \cos B)(\cos A - \sin B)(\cos A + \sin B)$

ii) $\cos^6 A - \sin^6 A = \cos 2A\left(1 - \frac{1}{4}\sin^2 2A\right)$

iii) $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$

(iv) $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A$

v) $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$.

vi) $\tan 37\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$

vii) $\cos 10^\circ \cdot \cos 30^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ = \frac{3}{16}$

viii) $\left(\dfrac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\dfrac{\sin A + \sin B}{\cos A - \cos B}\right)^n = 2\cot^n\left(\frac{A-B}{2}\right)$, if n is even

$= 0$, If n is odd.

(ix) $2\sin 105^\circ \cdot \sin 15^\circ = \frac{1}{2}$

(x) $\dfrac{1+\cos A}{\sin A} = \cot\left(\frac{A}{2}\right)$ and hence deduce that

$\cot 37\frac{1}{2}^\circ = \sqrt{6} - \sqrt{3} - \sqrt{2} + 2$

xi) $2\sin\frac{\pi}{32} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$

xii) If $\sin\theta + \operatorname{cosec}\theta = 2$ then show that $\sin^n\theta + \operatorname{cosec}^n\theta = 2$ for all +ve integers n

xiii) If $\cos A = \frac{1}{2}$, $\cos B = 1$ then prove that $\tan\left(\frac{A+B}{2}\right).\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$

xiv) $\sqrt{3}(3\tan 10° - \tan^3 10°) = 1 - 3\tan^2 10°$

20: If an Angle θ is divided into two parts α, β such that $\tan\alpha : \tan\beta = x : y$ then show that $\sin(\alpha-\beta) = \frac{x-y}{x+y}\sin\theta$.

21: Find

i) $\frac{\cos 15° + \sin 15°}{\cos 15° - \sin 15°}$ (ii) $\sin 105°.\cos 105°$ (iii) $2\sin 67\frac{1}{2}°.\cos 67\frac{1}{2}°$

(iv) $\frac{1-\tan 15°}{1+\tan 15°}$ (v) If $\tan x + \tan y = 5$, $\tan x.\tan y = \frac{1}{2}$ then find $\cot(x+y)$.

vi) $\cos 15°.\cos 7\frac{1}{2}°.\sin 7\frac{1}{2}°$ (vii) $\sin 20(3 - 4\cos^2 70°)$

viii) $2\tan 7\frac{1}{2}° \times \frac{1-\tan^2(7\frac{1}{2})°}{[1+\tan^2(7\frac{1}{2})°]^2}$ (ix) $\sin 35° + \cos 5°$

(x) $\sin^2 24° - \sin^2 6$

(22) If $\frac{1+\sin A}{\cos A} = \sqrt{2}+1$ then find $\frac{1-\sin A}{\cos A}$.

23) For what values of θ, $\cos 3\theta + \sin 3\theta$ is maximum.

24) Show that $\sin 10°.\sin 50°.\sin 60.\sin 70° = \frac{\sqrt{3}}{16}$

25) If $A+B+C = \pi$ then prove that

i) $\cos 2A + \cos 2B + \cos 2C + 1 + 4\cos A.\cos B.\cos C = 0$

(ii) $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A.\cos B.\cos C = 1$

(iii) $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A.\sin B.\sin C$.

(iv) $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right) = 4\cos\frac{\pi-A}{4}.\cos\frac{\pi-B}{4}.\cos\frac{\pi-C}{4}$

(v) $\cos 2A - \cos 2B - \cos 2C = -1 + 4\cos A.\sin B.\sin C$

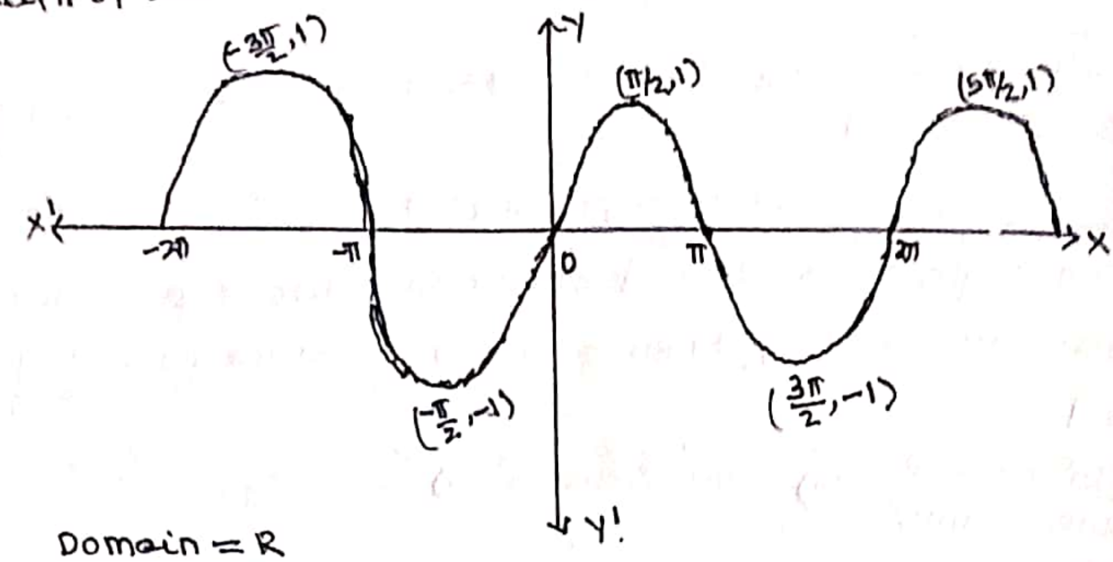
L: GRAPHS OF $\sin x$, $\cos x$, $\tan x$, e^x:

→ The graphs of a function is very useful way of visualizing the relationship of the function models and manipulating a mechanical expression for a function which can through light on the function's properties.

→ Graphs are used for modeling many different natural and mechanical phenomena (populations, waves, engine etc.)

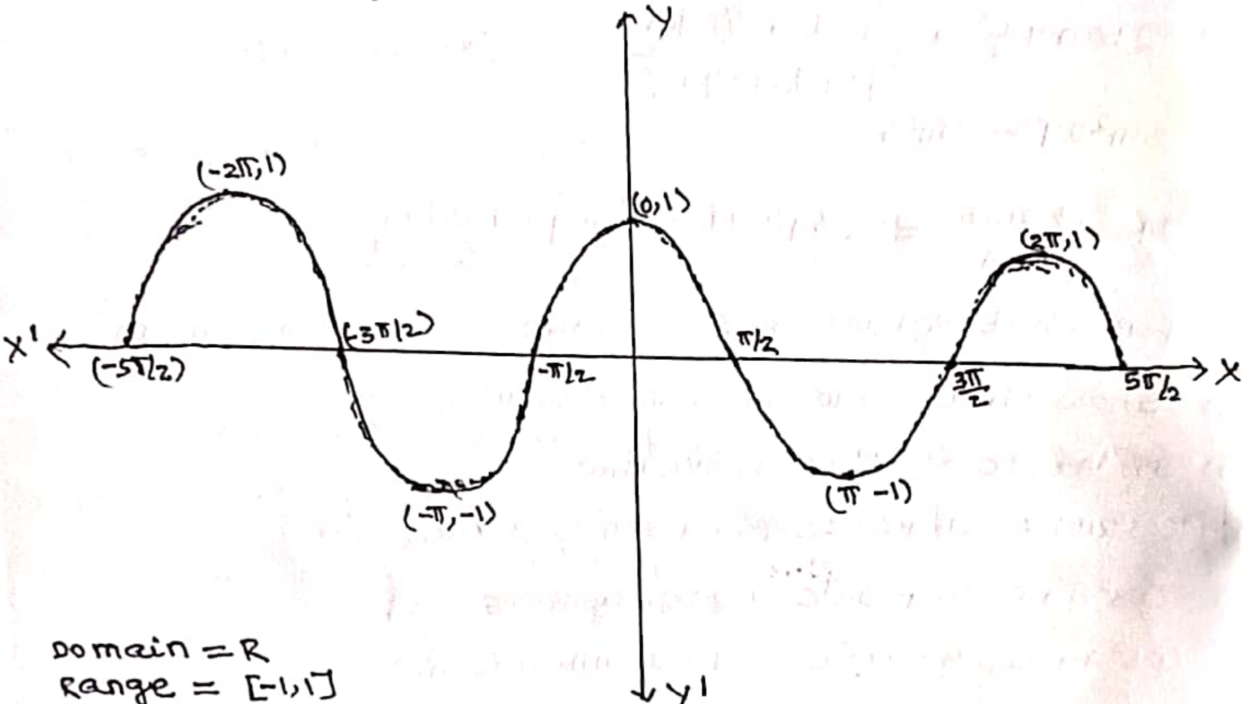
→ The trigonometric graphs are periodic i.e the shapes repeats ~~after~~ itself after certain amount of time.

(i) graph of $\sin x$



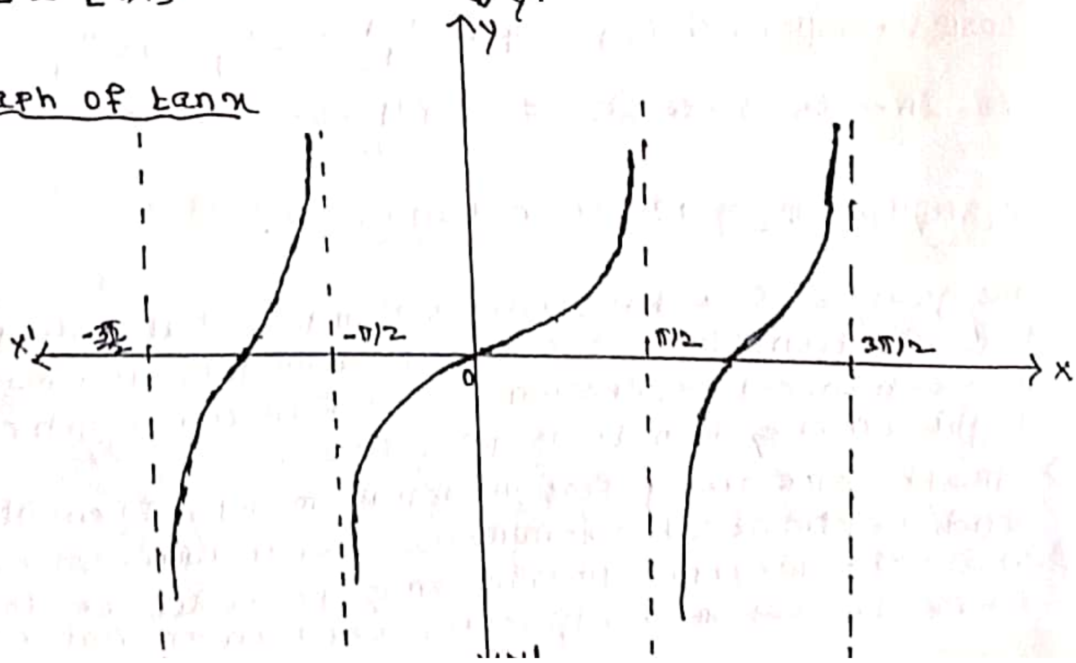
Domain = R
Range = [-1, 1]

(ii) graph of $\cos x$



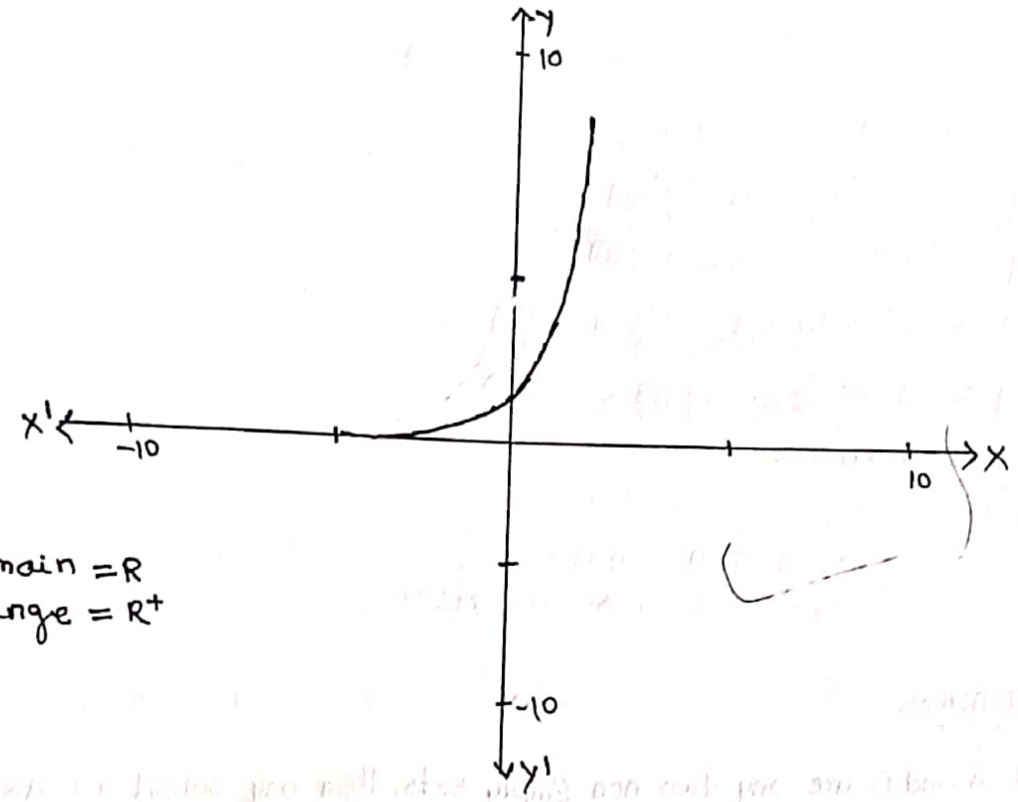
Domain = R
Range = [-1, 1]

(iii) Graph of $\tan x$



Domain $= R - \{(2n+1)\frac{\pi}{2}, n \in Z\}$, Range $= R$.

(iv) graph of e^x



Domain $= R$
Range $= R^+$

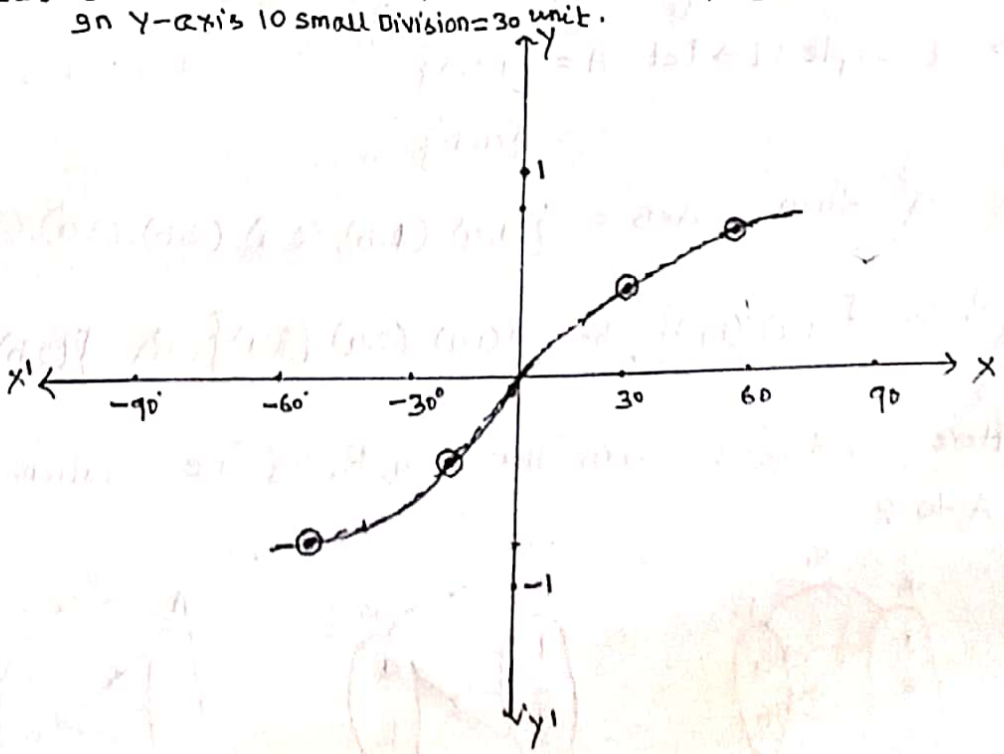
EX-41 : Draw the graph of $y = \sin x, -90° < x < 90°$

Soln: $y = \sin x, -90° < x < 90°$ ——(1)

Table:

x	$-60°$	$-30°$	$0°$	$30°$	$60°$
y	-0.86	-0.5	0	0.5	0.86.

Scale: In X-axis 10 small division = 30 unit.
In Y-axis 10 small Division = 30 unit.



HOME TASK-5:

Draw the graph of following:

(1) $y = 3\cos 2x, \; -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

(2) $y = \tan x, \quad -60^\circ \leq x \leq 60^\circ$

(3) $y = e^{2x}, \quad -1 \leq x \leq 1$

4) $y = \sin x, \quad 0 \leq x \leq 2\pi$

5) $y = 2\cos x, \quad \left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right)$

6) $y = 3\sin \frac{x}{2}, \quad (-2\pi < x < 2\pi)$

7) $y = \tan 2x$

UNIT-II, CHAPTER-2
FUNCTION AND LIMIT

A. RELATION:

→ If A and B are any two non-Empty sets, then any subset of $A \times B$ is Called a relation from A to B

ie If $R \subset A\times B \Rightarrow R$ is a relation from A to B

→ We know that $\phi \subset A\times B$ and $A\times B \subset A\times B$. Hence ϕ, $A\times B$ are also relations from A to B 'ϕ' is the Smallest relation and "$A\times B$" is the largest relation from A to B

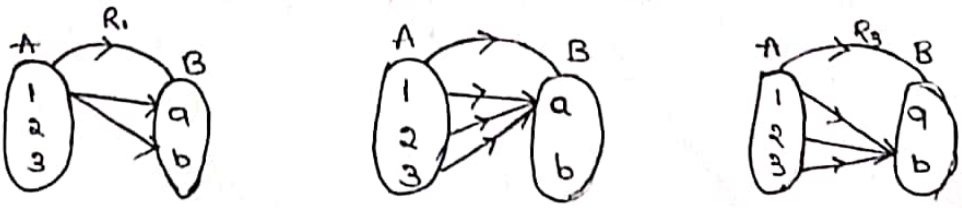
→ Example: 1→ Let $A = \{1,2,3\}$

$B = \{a, b\}$

then $A\times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

$R_1 = \{(1,a),(1,b)\}$, $R_2 = \{(1,a),(2,a),(3,a)\}$, $R_3 = \{(2,b),(3,b),(1,b)\}$

Here $R_1, R_2, R_3 \subset A\times B$. Hence R_1, R_2, R_3 are relations from A to B



→ If R is a relation from A to B then

$$\boxed{\begin{array}{l} \text{Domain of } R = \text{dom}.R = D_R = \{x \mid (x,y) \in R\} \\ \text{Range of } R = \text{Rng}.R = R_R = \{y \mid (x,y) \in R\} \end{array}}$$

In above example $D_{R_1} = \{1\}$, $R_{R_1} = \{a,b\}$

$D_{R_2} = \{1,2,3\}$, $R_{R_2} = \{a\}$

$D_{R_3} = \{1,2,3\}$, $R_{R_3} = \{b\}$

B. FUNCTION

A relation f from X to Y is said to be a function if

(a) $D_f = X$

(b) f is not one many

→ Hence every function must be a relation but the converse is not necessarily true.

→ If f is a function from X into Y such that $x \in X$ is related to $y \in Y$ then we write it as $f: X \to Y$ s.t $f(x) = y$ or $(x,y) \in f$ or xfy

Here y = image of x, x = preimage of y

X = domain of $f = D_f$

Y = Codomain of f = Codom f

→ Range of f is denoted by R_f and defined by

$$\boxed{R_f = \{y \in Y : y = f(x), x \in X\}}$$

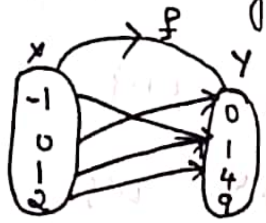
Clearly, $R_f \subseteq Y$

→ Example 2 :- Let $X = \{-1, 0, 1, 2\}$, $Y = \{0, 1, 4, 9\}$

$$f = \{(-1,1), (0,0), (1,1), (2,4)\}$$

Here f is a function from X into Y for

f is not one many and $D_f = X$



$D_f = \{-1, 0, 1, 2\} = X$, $R_f = \{0, 1, 4\}$, Codom $f = Y$

Clearly for $x \in X$, $\exists$ $y \in Y$ s.t $y = f(x) = x^2$

In Ex-1, R_1 is not a function, R_2 and R_3 are functions

→ If $f(x)$ is a function then $f(a)$ is Called functional value of $f(x)$ at $x = a$.

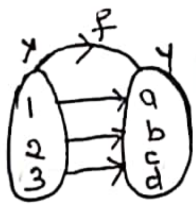
C : TYPES OF FUNCTION :

C_1 : into function :

A function $f : X \to Y$ is said to be into function if there exist at at least one element in Y which has no pre image in X

i.e for an into function f R_f is a proper subset of Y

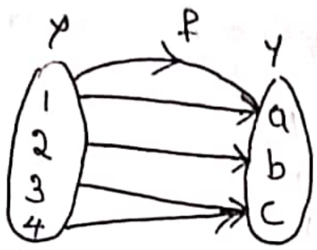
$f = \{(1,a), (2,b), (3,c)\}$ is an into function from X to Y



C_2 : onto function

A function $f : x \to y$ is said to be onto function if $R_f = Y$

$f = \{(1,a),(2,b),(3,c),(4,c)\}$ is an onto function from x to Y

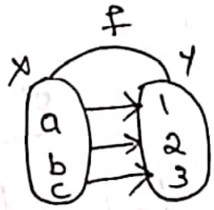


C_3 : one - one function

A function $f : x \to Y$ is said to be one-one function if each distinct element in x has distinct image in Y

i.e if $x_1, x_2 \in x$ s.t $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

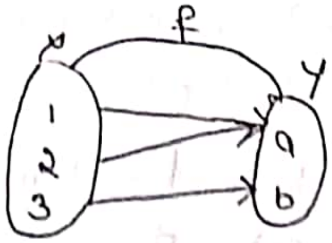
$f = \{(a,1),(b,2),(c,3)\}$ is an one-one function from x to y



C_4 : Many One function :

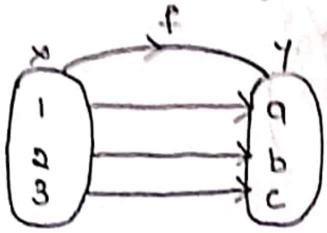
A function $f : x \to Y$ is said to be many one function if there exist at least one element in Y which has more than one Pre image in X

$f = \{(1,2),(2,a),(3,b)\}$ is a many-one function from x to y



C5 : Bijective function A function $f : X \rightarrow Y$ is said to be bijective If it is both onto and one-one i.e. each distinct element of x has distinct image in Y and every element of Y has preimage in X

$f = \{(1,a),(2,b),(3,c)\}$ is a bijective function from x to y



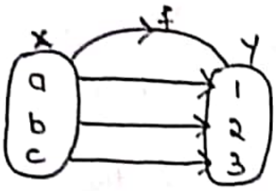
C6 : INVERSE OF A FUNCTION :

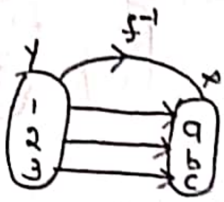
If $f : x \rightarrow Y$ is a bijective function then inverse of f is denoted by f^{-1} and defined by $f^{-1} : Y \rightarrow X$ s.t

$$y = f(x) \Rightarrow x = f^{-1}(y) \quad \forall\ x \in X,\ y \in Y$$

$f = \{(a,1),(b,2),(c,3)\}$ is bijective.

$f^{-1} = \{(1,a),(2,b),(3,c)\}$





C7: Composition of two function :

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be any two functions then Composition of f and g is denoted by gof and defined by $(gof) : X \rightarrow Z$ s.t $(gof)(x) = g(f(x))$, $\forall\ x \in X$

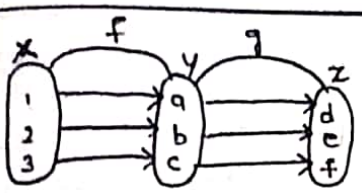
Example : Let $f = \{(1,a),(2,b),(3,c)\}$ is a function from x to Y

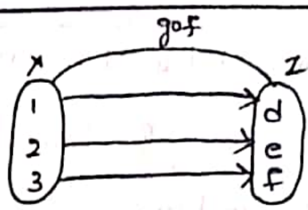
and $g = \{(a,d),(b,e),(c,f)\}$ is a function from Y to z

then (gof) : $= \{(1,d),(2,e),(3,f)\}$ is a function from x to z

for $(gof)(1) = g(f(1)) = g(a) = d$.

$(gof)(2) = e$, $(gof)(3) = f$





Example :

Let $f(x) = \sin x$ and $g(x) = x^2$

Now $(f \circ g)(x) = f(g(x)) = f(x^2) = \sin x^2$

$\therefore$ $\sin x^2$ is the Composition of x^2 and $\sin x$

C_8 : Real valued function :

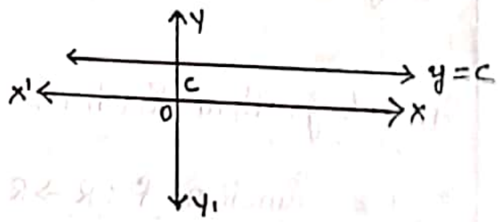
A function $f : X \rightarrow Y$ is Called a real valued function

If. $D_f = X \subset R$ and $Y \subset R$.

C_9 : CONSTANT FUNCTION : The function $f : R \rightarrow R$ defined by $f(x) = c$, $\forall\, x \in R$, $c \in R$ is a Constant, is Called Constant function

→ $D_f = R$, $R_f = \{c\}$

→ graph is parallel to x-axis.

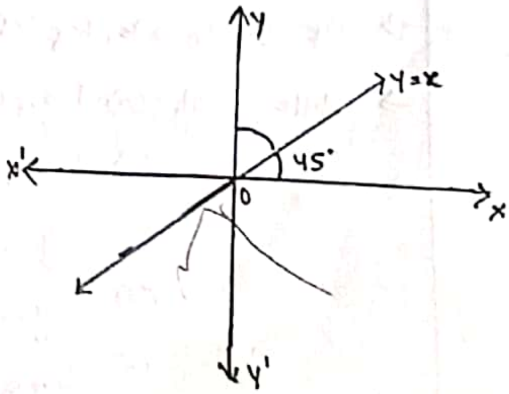


C_{10} : Identity function :

→ A function $f : R \rightarrow R$ defined by $f(x) = x$, $\forall\, x \in R$ is Called identity function

→ $D_f = R$, $R_f = R$

→ Its graph is a straight line passing through origin and bisecting the angle between the axes

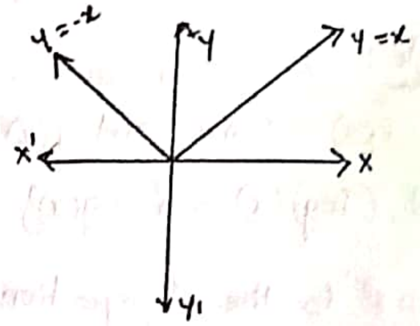


C11: Absolute Value/Modulus Function

→ The function $f: R \to R$ defined by $\boxed{f(x) = |x| = x,\ x \geq 0 \\ = -x,\ x < 0}$ is Called absolute value function

→ $D_f = R$, $R_f = R^+ = [0, \infty]$

→ Its graph is given in fig



C12: Signum Function: A function

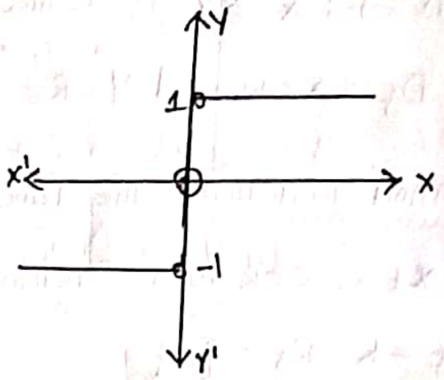
→ $f: R \to R$ defined by

$$\boxed{f(x) = \text{Sgn}(x) = \frac{|x|}{x},\ x \neq 0 \\ 0,\ x = 0}$$

is Called Signum function

→ $D_f = R$, $R_f = \{-1, 0, 1\}$

→ graph is given in fig.



C13: Logarithim Function:

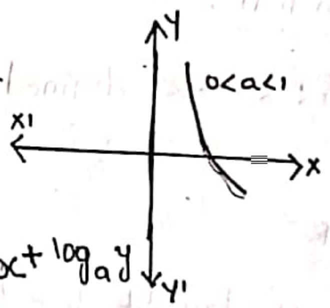
→ The function $f: R \to R$ defined by

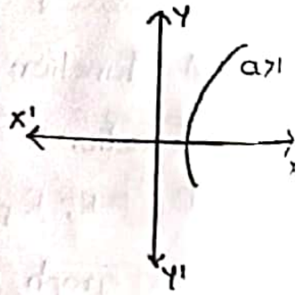
if $a^{f(x)} = x \Leftrightarrow f(x) = \log_a x$

Where $x > 0$, $a > 0$, $a \neq 1$, $\forall\ x, a \in R$

is Called logarithim function

→ $D_f = (0, \infty)$, $R_f = R$





→ Note that (a) $\log_a(xy) = \log_a x + \log_a y$

(b) $\log_a\left[\frac{x}{y}\right] = \log_a x - \log_a y$

(c) $\log_a x^n = n \log_a x$

(d) $\log_a a = 1$

(e) $\log_a x = 0 \Leftrightarrow x = 1$ i.e. $\log_a 1 = 0$

(f) $\log_a x = \frac{1}{\log_x a}$

(g) $\log_a x = \log_b x \times \log_a b$

(h) $e^{\ln x} = x$, $a^{\log_a x} = x$ (i) $\log_e x = \log x = \ln x$

C14: EXPONENTIAL FUNCTION

→ The function $f: R \to R$. defined by $f(x) = a^x, a>0, a\neq 1$, $a, x \in R$ is Called exponential function

→ $D_f = R$, $R_f = (0, \infty)$

→ Note that

(a) $a^x \cdot a^y = a^{x+y}$

(b) $\frac{a^x}{a^y} = a^{x-y}$

(c) $(a^x)^y = (a^y)^x = a^{xy}$

(d) $a^x \times b^x = [a \times b]^x$, (e) $\frac{a^x}{b^x} = \left[\frac{a}{b}\right]^x$

(f) $a^0 = 1, a \neq 0$ (g) $x^{-n} = \frac{1}{x^n}$, $\left[\frac{x}{y}\right]^{-n} = \left[\frac{y}{x}\right]^n$

(h) $a^x = a^y \Leftrightarrow x = y, a \neq 1$ (i) $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

C15: TRIGONOMETRIC FUNCTION:

sin x, cos x, tan x, cot x, sec x and Cosec x are Called trigonometric functions. These are real valued functions x is measured. in radian

FUNCTION	DOMAIN	RANGE
Sin x	R	[-1, 1]
Cos x	R	[-1, 1]
tan x	$R - \{(2n+1)\frac{\pi}{2} : n \in z\}$	R
Cot x	$R - \{n\pi : n \in z\}$	R
Sec x	$R - \{(2n+1)\frac{\pi}{2} : n \in z$	R - (-1, 1)
Cosec x	$R - \{n\pi : n \in z\}$	R - (-1, 1)

C_{16} : INVERSE TRIGONOMETRIC FUNCTION

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$ are called inverse trigonometric function which you have known

FUNCTION	DOMAIN	RANGE
$\sin^{-1}x$	$[-1,1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cot^{-1}x$	R	$(0,\pi)$
$\sec^{-1}x$	$[-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\{\frac{\pi}{2}\}$
$\operatorname{cosec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}]-\{0\}$

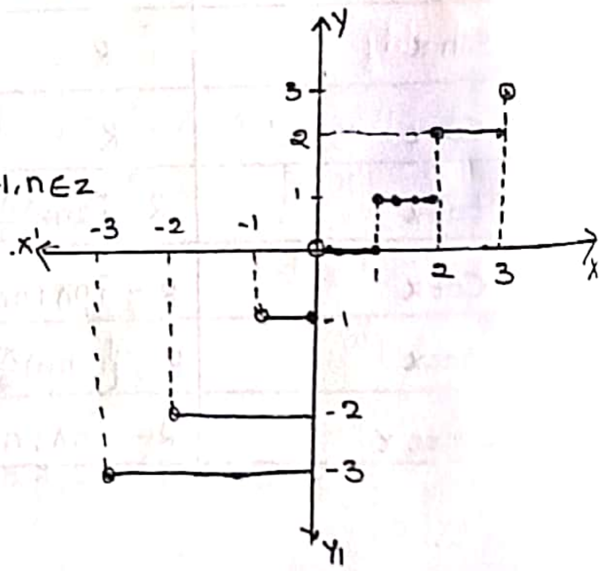
C_{17}: Greatest integer function [Bracket function)

→ For all $x \in R$, $\boxed{[x] = \text{Greatest integer} \le x}$ is called Greatest integer function

→ $D_f = R$, $R_f = z$

→ Thus

$$[x] = \begin{cases} x & \text{if } x \in Z \\ n & \text{If } x \notin z \text{ and } n < x < n+1, n \in z \end{cases}$$



→ Examples:

$[0] = 0$, $[1] = 1$, $[2] = 2$, $[-1] = -1$

$[-3] = -3$, $[-2] = -2$

$[2.5] = 2$ for $2 < 2.5 < 3$

$[-2.5] = -3$ for $-3 < -2.5 < -2$.

$[\sqrt{3}] = 1$ for $1 < \sqrt{3} < 2$

$[e] = 2$ for $2 < e < 3$

$[-e] = -3$ for $-3 < -e < -2$

$[\pi] = 3$, $[-\pi] = -4$

C_{18} : ALGEBRAIC FUNCTION

There are three types of algebraic function

(i) Polynomial function : $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Ex x^2+2x+3, $3x+5e$

(ii) Rational function $\frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n}$

Ex $\frac{x}{x^2+1}$, $\frac{x^2+2x+5}{3x+1}$

(iii) Irrational function : $P(x)^{\frac{p}{q}}$

Ex : $\sqrt{x}$, $[x^2+2x+1]^{2/3}$

C_{19}: TRANSCEENDENTAL FUNCTION:

Trigonometric, logarithmic, exponential function, inverse trigonometric functions are called transcendental functions.

C_{20}: EXPLICIT FUNCTION:

→ The function which can be expressed directly in terms of independent variable only is called explict function

→ $y = f(x)$ is the explict function

→ $y = \sin x$, $y = x^2+1$ etc.. are explicit function.

C_{21} : IMPLICIT FUNCTION :

→ The function in the form, $f(x,y) = 0$ i.e. in which x and y can't be separated from each other is called implicit function

→ Ex: $x^2+y^2=4$, $x^3+y^3-3axy=0$

→ An implicit function can be converted into explicit function whenever possible

Ex : $x^2+y^2=4$ (Implicit)

$\Rightarrow y = \sqrt{4-x^2}$ [Explicit]

C_{22}: Even Function :

→ A function $f(x)$ is said to be an even function if $\boxed{f(-x) = f(x) \; \forall x}$

→ $\cos x$ is an even function

for $f(x) = \cos x$

$\Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$ $(\because \cos(-\theta) = \cos\theta)$

Similarly $\sec x$, x^2, x^4 are even functions.

C_{23}: odd Function :

→ A Function $f(x)$ is said to be an odd function if

$$\boxed{f(-x) = -f(x) \; \forall x}$$

→ $\sin x$ is an odd function for

$f(x) = \sin x \Rightarrow f(-x) = \sin(-x) = -\sin x = -f(x)$

Similarly $\tan x$, $\cot x$, $\operatorname{cosec} x$, x^3 are odd function

D: NEIGHBOURHOOD : (nbd)

→ $(a-\delta, a+\delta)$ is Called nbd of $a \in R$, $\delta > 0$ is very very small quantity.

→ $(a-\delta, a)$ is Called left nbd of a

→ $(a, a+\delta)$ is Called right nbd of a

→ $(a-\delta, a+\delta) - \{a\}$ is Called deleted nbd of a

Ex: $(1.9, 2.1) \to$ nbd of 2

$(1.9, 2) \to$ left nbd of 2

$(2, 2.1) \to$ Right nbd of 2

$(1.9, 2.1) - \{2\} \to$ deleted nbd of 2.

E: Interval:

→ close interval: $[a,b] = \{x \mid a \le x \le b\}$

→ open interval: $(a,b) = \{x \mid a < x < b\}$

→ semi open or semiclose interval:

$[a,b) = \{x \mid a \le x < b\}$

$(a,b] = \{x \mid a < x \le b\}$.

F: METHODS FOR FINDING DOMAIN, RANGE:

→ Domain : • Expression under even root ≥ 0

• Denominator $\neq 0$

• $\text{Dom.}\{f(x) \pm g(x)\}, \text{Dom}\{f(x) \cdot g(x)\} = D_1 \cap D_2$

• $\text{Dom} \dfrac{f(x)}{g(x)} = D_1 \cap D_2 - \{x \mid g(x) = 0\}$

where $\text{dom} f(x) = D_1$, $\text{Dom} g(x) = D_2$.

→ Range: Range of $y = f(x)$ is the set of all outputs $f(x)$ corresponding to each real x in domain

- If domain consists of finite number of points, then range is the set of corresponding $f(x)$ values.
- If domain is a finite interval then find the least and greatest value of Range using monotonicity.

Ex-1: state giving reason, whether given relation is a function or not.

(i) $R = \{(2,1), (3,1), (4,2)\}$ (ii) $R = \{(2,2), (2,4), (3,3), (4,4)\}$ (iii) $R = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,7)\}$.

Soln:

i) $R = \{(2,1), (3,1), (4,2)\}$ is a function since ~~each~~ it is not one-many.

(ii) $R = \{(2,2), (2,4), (3,3), (4,4)\}$ is not a function since it is one many

(iii) $R = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,7)\}$ is a function as it is not one-many.

Ex-2: If $f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x < 1 \\ \frac{1}{x}, & x > 1 \end{cases}$ then find $f(\tfrac{1}{2}), f(-2), f(2), f(0)$.

Soln: $f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x < 1 \\ \frac{1}{x}, & x > 1 \end{cases}$ ——— (1)

$f(\tfrac{1}{2}) = \tfrac{1}{2} \quad \because 0 \le \tfrac{1}{2} < 1$

$f(-2) = (-2)^2 = 4 \quad \because -2 < 0$

$f(2) = \tfrac{1}{2} \quad \because 2 > 1$

$f(0) = 0 \quad \because 0 \le 0 < 1$

Ex-3: Find domain of $f(x) = \dfrac{x^2+3x+5}{x^2-5x+4}$.

Soln: $f(x) = \dfrac{x^2+3x+5}{x^2-5x+4}$

If $x^2-5x+4=0 \Rightarrow x^2-4x-x+4=0 \Rightarrow x(x-4)-1(x-4)=0$

$\Rightarrow (x-4)(x-1)=0 \Rightarrow x-4=0$ or $x-1=0 \Rightarrow x=4$ or $x=1$

Hence $f(x)$ is defined for all real x except $x=1,4$

So Dom. $f(x) = R - \{1,4\}$.

Ex-4 Find domain and Range of $f(x) = \sqrt{x-1}$.

Soln $f(x) = \sqrt{x-1}$

If $x-1 \geq 0 \Rightarrow x \geq 1$

$\therefore$ $f(x)$ is weldefined for all real $x \geq 1$

Hence $\text{dom}(f) = [1, \infty)$.

For each $y \geq 0$ in the co-domain set, $\exists$ x in domain set for which $y = f(x)$.

Hence Rng $f = [0, \infty)$.

G: LIMIT OF A FUNCTION:

G1: Indeterminate forms:

→ If the value of a functions do not have definite value to some perticular value of variable then such forms are called indeterminate forms.

→ Ex: $f(x) = \dfrac{x^2-9}{x-3}$, $f(3) = \dfrac{9-9}{3-3} = \dfrac{0}{0}$ which can't be determined.

→ Some indeterminate forms are $0\times\infty$, 0^0, 1^∞, $\infty \pm \infty$, $\dfrac{\infty}{\infty}$, $\infty\times\infty$, ∞^0, $\dfrac{0}{0}$

Similarly $\sec x, x^2, x^4$ are even functions.

C-2: Odd Function:

A function $f(x)$ is said to be an odd function if

$$\boxed{f(-x) = -f(x) \; \forall x}$$

→ $\sin x$ is an odd function for
$f(x) = \sin x \Rightarrow f(-x) = \sin(-x) = -\sin x = -f(x)$

similarly $\tan x, \cot x, \operatorname{cosec} x, x^3$ are odd function.

D: NEIGHBOURHOOD: (nbd.)

→ $(a-\delta, a+\delta)$ is called nbd of $a \in \mathbb{R}$, $\delta > 0$ is very very small quantity.

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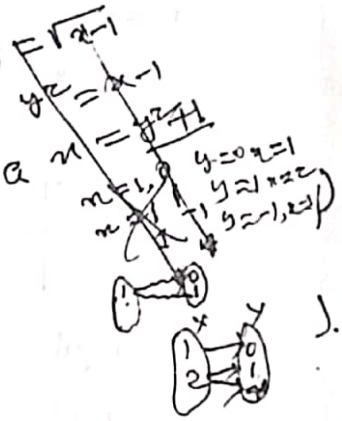

Ex $(1.9, 2.1) \rightarrow$ nbd. of 2

$(1.9, 2) \rightarrow$ Left nbd. of 2

$(2, 2.1) \rightarrow$ Right nbd. of 2

$(1.9, 2.1) - \{2\} \rightarrow$ deleted nbd. of 2.

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G-2: LIMIT OF A FUNCTION:

The concept of limit of a function is based on the notion of closeness or nearness. For better understanding the concept of limit of a function let's consider the function

$f(x) = y = 2x+1$ and let the independent variable x takes the values closer and closer to 2 from either sides of 2 i.e from left of 2 $(x<2)$ and right of 2 $(x>2)$. See following tables.

Table-1

x	2.1	2.01	2.001	2.0001
$f(x)$	5.2	5.02	5.002	5.0002

Table-2:-

x	1.9	1.99	1.999	1.9999
$f(x)$	4.8	4.98	4.998	4.9998

From table ① we see that when x approaches towards 2 from right of 2 $(x \to 2^+)$, $f(x)$ approaches towards 5 $(f(x) \to 5)$. In this situation. We can say the limiting value of $f(x)$ is 5. Symbollyically. $\lim_{x \to 2^+} f(x) = 5$
This limit is called right hand limit of $f(x)$ at $x=2$.

From table (2) we see that when x approaches towords 2 from left of 2 ($x \to 2^-$) $f(x)$ approaches towards 5 ($f(x) \to 5$). In this situation we can say that the limiting value of $f(x)$ is 5 Symbollically $\lim_{x \to 2^-} f(x) = 5$. This limit is called left hand limit of $f(x)$ at $x=2$.

Combining above two situation we can say when $x \to 2$, $f(x) \to 5$.

ie. $\lim f(x) = 5$. This is called limit of $f(x)$ at $x=2$.

$\therefore \lim_{x \to 2} f(x) = \lim_{x \to 2} (2x+1) = 2.2+1$ [by putting direct by $x=2$]

$= 5$

Let's consider another function $f(x) = \frac{x^2-9}{x-3}$ at $x=3$. This function is not defined at $x=3$ $[\frac{0}{0}]$ But when x approaches towards 3 from both sides of 3, $f(x)$ approaches towards 6.

Table-3:-

x	2.9	2.99	2.999	3.1	3.01	3.001
$f(x)$	5.9	5.99	5.999	6.1	6.01	6.001

From above table we see that $\lim_{x \to 3^-} f(x) = 6$.

$\lim_{x \to 3^-} f(x) = 6$. Hence $\lim_{x \to 3} f(x) = 6$. Thus while calculating limit of a function we can't substitute $x=a$ (if $x \to a$) directly in $f(x)$ always until it is not a finite number.

Hence $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{[x^2-9]}{(x-3)}$ $(\frac{0}{0})$

$= \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x \to 3} (x+3) = 3+3 = 6$

From above examples we may conclude followings

(i) Given $\epsilon > 0$, there exist $\delta > 0$. Such that if $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon$ then l is called limit of $f(x)$ at $x=a$. Mathematically,

$\lim_{x \to a} f(x) = l$

Explanation :- From table 3 we observe that when $x-3$ decreases $f(x)-6$ also decreases.

Hence by making $|x-3|$ sufficiently small, $|f(x)-6|$ can be made as small as passible. Let $|f(x)-6| < 0.0001$ (ϵ).

$\Rightarrow \left|\frac{x^2-9}{x-3} - 6\right| < 0.0001 \Rightarrow |x+3-6| < 0.0001$

$\Rightarrow |x-3| < 0.0001$ $(\delta = 0.0001)$ Hence we observe that there exist a positive number δ. Such that $|x-3| < \delta \Rightarrow |f(x)-6| < \epsilon$

(ii) When x approaches from Left of a than the value to which $f(x)$ approaches is Called Left hand limit of $f(x)$ at $x=a$ written as LHL

$$\boxed{LHL = \lim_{x\to a^-} f(x) = \lim_{h\to 0^+} f(a-h),\ h>0}$$

(iii) When x approaches from right of a then the value to which $f(x)$ approaches is called Right hand limit of $f(x)$ at $x=a$ written as RHL

$$\boxed{RHL = \lim_{x\to a^+} f(x) = \lim_{h\to 0^+} f(a+h),\ h>0}$$

NOTE: $\lim_{x\to a^+} f(x) = \lim_{x\to a+0} f(x) = f(a+0)$

$\lim_{x\to a^-} f(x) = \lim_{x\to a-0} f(x) = f(a-0)$.

(iv) $\lim_{x\to a} f(x)$ exist if (a) RHL and LHL both exist and (b) RHL = LHL

(v) $x \to a$ (x tends to a) (vi) $x\to a \Rightarrow x \neq a$ i.e $x<a$ or $x>a$ (slightly)

(vii) $x \to a^+ \Rightarrow x \neq a$ and $x>a$ (slightly) (viii) $x \to a^- \Rightarrow x \neq a$ and $x<a$ (slightly).

(ix) The approximate value of a function Corresponding to approximate value of independent variable is called limit of a function.

Q.3) ALGEBRA OF LIMIT: Let $f(x)$ and $g(x)$ are any two functions and c be a Constant. Then

(i) $\lim_{x\to a} \{f(x) \pm g(x)\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a} \{f(x)\cdot g(x)\} = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\ g(a) \neq 0$

(iv) $\lim_{x\to a} [c.f(x)] = c.\lim_{x\to a} f(x)$

(v) $\lim_{x\to a} (c) = c$

vi) $\lim_{x\to a} f(g(x)) = f.\left[\lim_{x\to a} g(x)\right]$

vii) $\lim_{x\to a} f(x) = \lim_{y\to f(a)} y$

G. USEFUL STANDARD FORMULAE:-

(i) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$, a = Constant

(ii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a = \log a = \ln a$

(iii) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(iv) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\ln a}$

(v) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(vi) $\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

(vii) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$

$\lim_{x\to 0} \frac{\sin^{-1}x}{x} = 1$

$\lim_{x\to 0} \frac{\tan^{-1}x}{x} = 1$

(viii) $\lim_{x\to\infty} \frac{1}{x^n} = 0$

(ix) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} (1+f(x))^{g(x)} = e^{\lim_{x\to a} f(x) g(x)}$

H. METHODS OF EVALUATION :

(i) By direct Substitution:- To evaluate $\lim_{x\to a} f(x)$, put $x = a$ in $f(x)$ if. $f(a)$ is a finite number, that is if $f(x)$ doesn't attain indeterminate form $\left(\frac{0}{0}, \frac{1}{0}, 1^\infty, \frac{\infty}{\infty}, \infty\times\infty \ldots\right)$

(ii) Factorisation:- By putting $x=a$, if $f(a)$ is not a finite number the apply factorisation using algebraic formula. if possible

(iii) Rationalisation:- If direct substitution fails and N^r or D^r or both of the function Contains irrational function (functions involving square root sign) then apply rationalisation.

(iv) Using standard formula:- If direct Substitution fails then if possible Convert the given function into any standard form given in Section (G) and apply that Standard formula

(v) Limit when $x \to \infty$: In this case

(a) Keep the function in the form of $\frac{f(x)}{g(x)}$

(b) Divide N^r and D^r by x^n where n is the highest power among N^r and D^r.

(c) If $f(x)$ or $g(x)$ or both contains square root sign then apply the method 'Limits when $x \to \infty$' or apply rationalization first and then 'Limits when $x \to \infty$" which is applicable.

(vi) By sandwich theorem: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$ and $f(x) \le h(x) \le g(x)$ then $\lim_{x \to a} h(x) = L$.

5) Find $\lim_{x\to 4^+} \frac{|x-4|}{x-4}$

Sol^n $\lim_{x\to 4^+} \frac{|x-4|}{(x-4)} = \lim_{h\to 0^+} \frac{|(4+h)-4|}{(4+h)-4} = \lim_{h\to 0^+} \frac{|h|}{h}$

$= \lim_{h\to 0^+} \left(\frac{h}{h}\right)$ $(\because h\to 0^+ \Rightarrow h>0 \Rightarrow |h|=h)$

$= \lim_{h\to 0^+} (1) = 1$

6) Find left hand limit of $[x]$ at $x=3$

Sol^n LHL $= \lim [x] = \lim_{h\to 0^+} [3-h] = 2$

$(\because 2 < 3-h < 3)$

7) Show that $\lim_{x\to K} [x]$, $K\in Z$ doesnot exist.

Sol^n Let $f(x) = [x]$

RHL $= \lim_{x\to K^+} f(x) = \lim_{x\to K^+} [x] = \lim_{h\to 0^+} [K+h]$

$= K$ (exist) $(\because K < K+h < K+1)$

LHL $= \lim_{x\to K^-} f(x) = \lim_{x\to K^-} [x] = \lim_{h\to 0^+} [K-h]$

$= K-1$ (exist) $(\because K-1 < K-h < K)$

$\therefore$ RHL, LHL both exist but RHL $\neq$ LHL

Hence $\lim_{x\to K} [x]$, $K\in Z$ doesnot exist

8) Test the existence of $\lim f(x)$ where

$$f(x) = \begin{cases} 5x-4, & 0 < x \leq 1 \\ 4x^3-3x, & 1 < x < 2. \end{cases}$$

Sol^n
$$f(x) = \left.\begin{cases} 5x-4, & 0 < x \leq 1 \\ 4x^3-3x, & 1 < x < 2 \end{cases}\right\} \text{— (1)}$$

RHL $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x^3-3x) = \lim_{h\to 0^+} \left[4(1+h)^3 - 3(1+h)\right]$

$= 4(1+0)^3 - 3(1+0) = 4-3 = 1$ (exist)

$LHL = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5x-4) = \lim_{h\to 0^+} [5(1-h)-4] = 5(1-0)-4$
$= 5-4 = 1$ (exist)

$\therefore$ RHL, LHL both exist and RHL = LHL = 1. Hence $\lim_{x\to 1} f(x)$ exist and $\lim_{x\to 1} f(x) = 1$.

9) Show that $\lim_{x\to 0} \frac{|x|}{x}$ doesnot exist.

Soln Let $f(x) = \frac{|x|}{x}$

$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{h\to 0^+} \frac{|0+h|}{0+h} = \lim_{h\to 0^+} \frac{|h|}{h}$

$= \lim_{h\to 0^+} \left(\frac{h}{h}\right) = \lim_{h\to 0^+} (1) = 1$ (exist) $\quad \because h\to 0^+ \Rightarrow h>0$

$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{h\to 0^+} \frac{|0-h|}{0-h} = \lim_{h\to 0^+} \frac{|-h|}{-h} = \lim_{h\to 0^+} \frac{|h|}{(-h)} =$

$\lim_{h\to 0^+} \left(\frac{h}{-h}\right) = \lim_{h\to 0^+} (-1) = -1$ (exist)

$\therefore$ RHL, LHL both exist but RHL $\neq$ LHL. Hence $\lim_{x\to 0} \frac{|x|}{x}$ doesnot exist.

10) Evaluate $\lim_{x\to 0} \left(\frac{\cos x}{1+\sin x}\right)$

Soln $\lim_{x\to 0} \left(\frac{\cos x}{1+\sin x}\right) = \frac{\lim_{x\to 0} (\cos x)}{\lim_{x\to 0} (1+\sin x)} = \frac{\cos 0}{1+\sin 0} = \frac{1}{1+0}$

$= 1$

11) Evaluate $\lim_{x\to 2} \frac{x^3-6x^2+11x-6}{x-2}$

Soln $\lim_{x\to 2} \frac{x^3-6x^2+11x-6}{x-2} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 2} \frac{(x-2)(x^2-4x+3)}{(x-2)}$

$= \lim_{x\to 2} (x^2-4x+3) \quad (\because x\to 2 \Rightarrow x\neq 2 \Rightarrow x-2\neq 0)$

$= 2^2 - 4\times 2 + 3 = -1$

Rough

$x-2 \,)\, x^3-6x^2+11x-6 \,(\, x^2-4x+3$
$x^3 - 2x^2$ ($-$, $+$)
$-4x^2+11x-6$
$-4x^2 + 8x$ ($+$, $-$)
$3x-6$
$3x-6$ ($-$, $+$)
0

12) $\lim_{x\to 2} \frac{x^2-5x+6}{x^2-6x+8}$

Sol^n $\lim_{x\to 2} \frac{x^2-5x+6}{x^2-6x+8} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 2} \frac{x^2-3x-2x+6}{x^2-4x-2x+8} = \lim_{x\to 2} \frac{x(x-3)-2(x-3)}{x(x-4)-2(x-4)}$

$= \lim_{x\to 2} \frac{(x-3)(x-2)}{(x-4)(x-2)} = \lim_{x\to 2} \left[\frac{x-3}{x-4}\right] \quad \because x\to 2 \Rightarrow x\neq 2$

$\Rightarrow x-2\neq 0.$

$= \frac{2-3}{2-4} = \frac{-1}{-2} = \frac{1}{2}$

13) Evaluate $\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

Sol^n $\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to a} \frac{\left[\sqrt{a+2x}-\sqrt{3x}\right]\times\left[\sqrt{3a+x}+2\sqrt{x}\right]\times\left[\sqrt{a+2x}+\sqrt{3x}\right]}{\left[\sqrt{3a+x}-2\sqrt{x}\right]\times\left[\sqrt{3a+x}+2\sqrt{x}\right]\times\left[\sqrt{a+2x}+\sqrt{3x}\right]}$

$= \lim_{x\to a} \frac{\left\{(\sqrt{a+2x})^2-(\sqrt{3x})^2\right\}\times\left[\sqrt{3a+x}+2\sqrt{x}\right]}{\left\{(\sqrt{3a+x})^2-(2\sqrt{x})^2\right\}\times\left[\sqrt{a+2x}+\sqrt{3x}\right]}$

$\lim_{x\to a} \frac{(a-x)\left[\sqrt{3a+x}+2\sqrt{x}\right]}{3(a-x)\left[\sqrt{a+2x}+\sqrt{3x}\right]}$

$= \lim_{x\to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3\left[\sqrt{a+2x}+\sqrt{3x}\right]} \quad (\because x\to a \Rightarrow x\neq a \Rightarrow x-a\neq 0)$

$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3\left[\sqrt{a+2a}+\sqrt{3a}\right]} = \frac{4\sqrt{a}}{6\sqrt{3a}} = \frac{4\times\sqrt{a}}{6\times\sqrt{3}\times\sqrt{a}} = \frac{2}{3\sqrt{3}}$

14) Evaluate $\lim_{x\to\infty} \frac{3x^2+5x+7}{4x^2+8x-9}$

Soln $\lim_{x\to\infty} \frac{3x^2+5x+7}{4x^2+8x-9} = \lim_{x\to\infty} \frac{3+\frac{5}{x}+\frac{7}{x}}{4+\frac{8}{x}-\frac{9}{x}}$

(Dividing N^r & D^r by x^2)

$= \frac{3+0+0}{4+0+0} = \frac{3}{4}$

15) Evaluate $\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right)$

Soln $\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right.$ $(\infty-\infty)$

$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x+1}-\sqrt{x^2+1}\right)\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}$

$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x+1}\right)^2 - \left(\sqrt{x^2+1}\right)^2}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)} = \lim_{x\to\infty} \frac{x^2+x+1-x^2-1}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$

$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$

$= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+x\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}}$

$= \frac{1}{\sqrt{1+0+0}+\sqrt{1+0}} = \frac{1}{2}$

16) Evaluate $\lim_{n\to\infty} \frac{1^3+2^3+3^3+\cdots\cdots+n^3}{n^4}$

Soln $\lim_{n\to\infty} \frac{1^3+2^3+3^3+\cdots\cdots+n^3}{n^4} = \lim_{n\to\infty} \frac{\left[\frac{n(n+1)}{2}\right]^2}{n^4} = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4}$

$= \frac{1}{4} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{1}{4} \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{4} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2$

$= \frac{1}{4}(1+0)^2 = \frac{1}{4}$

17) Evaluate $\lim_{x\to\infty} x \sin\frac{1}{x}$

Soln: $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$

$= \lim_{\theta\to 0} \frac{1}{\theta} \sin\theta$

$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

[put $\frac{1}{x} = \theta$
Then when $x\to\infty, \theta\to 0$
and $x = \frac{1}{\theta}$

18) Evaluate $\lim_{x\to 0} \frac{(x+9)^{3/2} - 27}{x}$

Soln $\lim_{x\to 0} \frac{(x+9)^{3/2} - 27}{x}$

put $x+9 = y$
when $x\to 0 \Rightarrow y\to 9$
$\Delta\, x = y-9$

$= \lim_{y\to 9} \frac{y^{3/2} - 27}{y-9}$

$= \lim_{y\to 9} \frac{y^{3/2} - (9)^{3/2}}{y-9} = \frac{3}{2}\cdot(9)^{\frac{3}{2}-1}$

$= \frac{3}{2}\times 9^{\frac{1}{2}} = \frac{3}{2}\times(3^2)^{\frac{1}{2}} = \frac{3}{2}\times 3 = \frac{9}{2}.$

19) Evaluate $\lim_{x\to -1} \frac{\log(3x+4)}{x+1}$

Soln $\lim_{x\to -1} \frac{\log(3x+4)}{x+1} \quad \left(\frac{0}{0}\right)$

put $x+1 = y$
when $x\to -1, y\to 0$
$\Delta\, x = y-1$

$= \lim_{y\to 0} \frac{\log[3(y-1)+4]}{y}$

$= \lim_{y\to 0} \frac{\log(3y+1)}{y}$

$= \lim_{y\to 0} \left[\frac{\log(3y+1)}{3y}\times 3\right] = 3\times \lim_{3y\to 0} \frac{\log(3y+1)}{3y} \quad \because y\to 0,\ 3y\to 0$

$= 3\times 1 = 3.$

20) Evaluate $\lim_{x\to 0} \frac{a^x - b^x}{x}$

Soln $\lim_{x\to 0} \frac{a^x-b^x}{x} = \lim_{x\to 0} \frac{a^x-1+1-b^x}{x} = \lim_{x\to 0} \frac{(a^x-1)-(b^x-1)}{x}$

$= \lim_{x\to 0} \left\{\frac{a^x-1}{x} - \frac{b^x-1}{x}\right\} = \lim_{x\to 0} \frac{a^x-1}{x} - \lim_{x\to 0} \frac{b^x-1}{x}$

$= \ln a - \ln b = \ln\left[\frac{a}{b}\right]$

21) Evaluate $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$.

Soln $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}$

$= \frac{a}{b} \lim_{x\to 0} \frac{\frac{\sin ax}{ax}}{\frac{\sin bx}{bx}} = \frac{a}{b} \frac{\lim_{ax\to 0} \frac{\sin ax}{ax}}{\lim_{bx\to 0} \frac{\sin bx}{bx}}$.

$= \frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b}$

22) Evaluate $\lim_{x\to 0} \frac{\sin x^\circ}{x}$

Soln: $\lim_{x\to 0} \left[\frac{\sin x^\circ}{x}\right] = \lim_{x\to 0} \frac{\sin\left(x \times \frac{\pi}{180}\right)}{x}$ $\quad \because x^\circ = \left(x \times \frac{\pi}{180}\right)^c$

$= \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x} = \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}} \times \left(\frac{\pi}{180}\right)$

$= \frac{\pi}{180} \times \lim_{\frac{\pi x}{180}\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{\left(\frac{\pi x}{180}\right)} = \frac{\pi}{180} \times 1 = \frac{\pi}{180}$ $\quad (= 1^\circ)$

23) Evaluate $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$

Soln: $\lim_{\theta\to 0} \left(\frac{\tan\theta}{\theta}\right) = \lim_{\theta\to 0} \left(\frac{\sin\theta}{\cos\theta} \times \frac{1}{\theta}\right)$.

$= \lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}\right) = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \times \lim_{\theta\to 0} \frac{1}{\cos\theta}$

$= 1 \times \frac{1}{1} = 1$

24 Evaluate $\lim_{x\to 0} \frac{\tan^{-1}x}{x}$

Soln: $\lim_{x\to 0} \frac{\tan^{-1}x}{x} = \lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = \lim_{\theta\to 0}\left[\frac{\theta}{\frac{\sin\theta}{\cos\theta}}\right]$ [Put $\tan^{-1}x = \theta$ $\Rightarrow x = \tan\theta$ when $x\to 0, \theta\to 0$

$= \lim_{\theta\to 0}\left(\theta \times \frac{\cos\theta}{\sin\theta}\right) = \lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta} \times \cos\theta\right)$

$= \lim_{\theta\to 0} \frac{\theta}{\sin\theta} \times \lim_{\theta\to 0} \cos\theta = \lim_{\theta\to 0} \frac{1}{(\sin\theta/\theta)} \times 1 = \frac{1}{1} = 1$

25) Evaluate $\lim_{x\to 0}\left[\frac{x - x\cos 2x}{\sin^3 2x}\right]$

Soln: $\lim_{x\to 0}\left[\frac{x - x\cos 2x}{\sin^3 2x}\right] = \lim_{x\to 0} \frac{x[1-\cos 2x]}{(\sin 2x)^3}$

$= \lim_{x\to 0} \frac{x[2\sin^2 x]}{(2\sin x\cdot\cos x)^3} = \lim_{x\to 0} \frac{2x\sin^2 x}{8\sin^3 x\cdot\cos^3 x}$

$= \frac{1}{4} \lim_{x\to 0}\left(\frac{x}{\sin x\cdot\cos^3 x}\right)$

$= \frac{1}{4}\cdot \lim_{x\to 0}\left(\frac{x}{\sin x} \times \frac{1}{\cos^3 x}\right)$

$= \frac{1}{4}\cdot \lim_{x\to 0} \frac{1}{\frac{\sin(x)}{x}} \times \lim_{x\to 0} \frac{1}{\cos^3 x}$

$= \frac{1}{4} \times \frac{1}{1} \times \frac{1}{1^3} = \frac{1}{4}$

26) Find a if $\lim_{x\to 1} \frac{5^x - 5}{(x-1)\log_e a} = 5$

Soln $\lim_{x\to 1} \frac{5^x - 5}{(x-1)\log_e a} = 5$

$\Rightarrow \frac{1}{\log_e a}\cdot \lim_{x\to 1} \frac{5^x - 5}{(x-1)} = 5$

$\Rightarrow \frac{1}{\log_e a} \lim_{x\to 1} \frac{5(5^{x-1} - 1)}{x-1} = 5 \Rightarrow \frac{5}{\log_e a} \lim_{x-1\to 0} \frac{5^{x-1} - 1}{x-1}$

$\Rightarrow \frac{5}{\log_e a} \times \log 5 = 5 \Rightarrow 5\log a = 5\log 5$

$\Rightarrow a = 5.$

27: If $f(x) = \begin{cases} x^2+2, & x \geq 1 \\ 2x+1, & x < 1 \end{cases}$ then find $\lim_{x\to 1} f(x)$

Sol^n: $f(x) = \begin{cases} x^2+2, & x \geq 1 \\ 2x+1, & x < 1 \end{cases}$ —— (1)

$RHL = \lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} f(1+h) = \lim_{h\to 0^+} \{(1+h)^2+2\} = (1+0)^2+2 = 3$

$LHL = \lim_{x\to 1^-} f(x) = \lim_{h\to 0^-} f(1-h) = \lim_{h\to 0^+} \{2(1-h)+1\} = 2(1-0)+1 = 3$

RHL, LHL both exist and $RHL = LHL = 3$

$\therefore \lim_{x\to 1} f(x)$ exist and $\lim_{x\to 1} f(x) = 3$.

28: Estimate: $\lim_{x\to\sqrt{2}} \frac{x^4-4}{x-\sqrt{2}}$

Sol^n: $\lim_{x\to\sqrt{2}} \frac{x^4-4}{x-\sqrt{2}} = \lim_{x\to\sqrt{2}} \frac{x^4-(\sqrt{2})^4}{x-\sqrt{2}} = 4(\sqrt{2})^{4-1} = 4\times(2^{\frac{1}{2}})^3$

$= 4\times 2^{\frac{3}{2}} = 2^2\times 2^{\frac{3}{2}} = 2^{2+\frac{3}{2}} = 2^{\frac{7}{2}} = \sqrt{2^7} = 8\sqrt{2}$ (Ans)

29. Evaluate $\lim_{x\to -1} \frac{x^{17}+1}{x^9+1}$

Sol^n $\lim_{x\to -1} \frac{x^{17}+1}{x^9+1} = \lim_{x\to -1} \dfrac{\frac{x^{17}+1}{x+1}}{\frac{x^9+1}{x+1}} = \lim_{x\to -1} \frac{\{x^{17}-(-1)^{17}\}/\{x-(-1)\}}{\{x^9-(-1)^9\}/\{x-(-1)\}}$

$= \lim_{x\to -1} \frac{x^{17}-(-1)^{17}}{x-(-1)} \Big/ \lim_{x\to -1} \frac{x^9-(-1)^9}{x-(-1)} = \frac{17\times(-1)^{17-1}}{9\times(-1)^{9-1}} = \frac{17\times(-1)^{16}}{9\times(-1)^8}$

$= \frac{17\times 1}{9\times 1} = \frac{17}{9}$ (Ans)

30. Evaluate: $\lim_{x\to 0} \frac{5x\cos x - 2\sin x}{7x + 5\sin x}$

Sol^n: $\lim_{x\to 0} \frac{5x\cos x - 2\sin x}{7x + 5\sin x} = \lim_{x\to 0} \frac{5\cos x - 2\frac{\sin x}{x}}{7 + \frac{5\sin x}{x}}$

(Dividing Nr and Dr by x)

$= \dfrac{5\lim_{x\to 0}\cos x - 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)}{\lim_{x\to 0} 7 + 5\lim_{x\to 0}\left(\frac{\sin x}{x}\right)} = \frac{5\times 1 - 2\times 1}{7+5\times 1} = \frac{3}{12} = \frac{1}{4}$ (Ans)

31: Evaluate $\lim_{x\to 0} \frac{1-\cos 2x}{x}$

Sol^n: $\lim_{x\to 0} \frac{1-\cos 2x}{x} = \lim_{x\to 0} \frac{2\sin^2 x}{x} = 2\cdot\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\cdot x\right)$

$= 2\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0} x = 2\times 1^2\times 0 = 0$

32: Evaluate: $\lim_{\theta\to 0} \frac{\operatorname{cosec}\theta - \cot\theta}{\theta}$

Soln: $\lim_{\theta\to 0} \frac{\operatorname{cosec}\theta-\cot\theta}{\theta} = \lim_{\theta\to 0} \frac{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}}{\theta} = \lim_{\theta\to 0} \frac{\frac{1-\cos\theta}{\sin\theta}}{\theta}$

$= \lim_{\theta\to 0} \left(\frac{1-\cos\theta}{\theta\cdot\sin\theta}\right) = \lim_{\theta\to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{\theta\sin\theta(1+\cos\theta)} = \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta\cdot\sin\theta(1+\cos\theta)}$

$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta\cdot\sin\theta\cdot(1+\cos\theta)} = \lim_{\theta\to 0} \left\{\frac{\sin\theta}{\theta} \times \frac{1}{(1+\cos\theta)}\right\}$

$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \times \lim_{\theta\to 0} \frac{1}{1+\cos\theta} = 1 \times \frac{1}{1+1} = \frac{1}{2}$ (Ans).

33. Evaluate $\lim_{x\to 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x}$

Soln: $\lim_{x\to 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x} = \lim_{x\to 0} \frac{\frac{1}{\cos 5x} - \frac{1}{\cos 3x}}{\frac{1}{\cos 3x} - \frac{1}{\cos x}}$

$= \lim_{x\to 0} \left(\frac{\cos 3x - \cos 5x}{\cos 5x\cdot\cos 3x} \times \frac{\cos 3x\cdot\cos x}{\cos x - \cos 3x}\right)$

$= \lim_{x\to 0} \left(\frac{\cos 3x - \cos 5x}{\cos x - \cos 3x} \times \frac{\cos x}{\cos 5x}\right) = \lim_{x\to 0} \frac{\cos 3x - \cos 5x}{\cos x - \cos 3x} \times \lim_{x\to 0} \frac{\cos x}{\cos 5x}$

$= \lim_{x\to 0} \frac{-2\sin\left(\frac{3x+5x}{2}\right).\sin\left(\frac{3x-5x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right).\sin\left(\frac{x-3x}{2}\right)} \times \frac{1}{1} = \lim_{x\to 0} \frac{\sin 4x\cdot\sin x}{\sin 2x\cdot\sin x}$

$= \lim_{x\to 0} \frac{\sin 4x}{\sin 2x} = \lim_{x\to 0} \frac{\frac{\sin 4x}{4x}\times 4x}{\frac{\sin 2x}{2x}\times 2x} = 2\cdot \frac{\lim_{4x\to 0} \frac{\sin 4x}{4x}}{\lim_{2x\to 0} \frac{\sin 2x}{2x}}$

$= 2\times\frac{1}{1} = 2$ (Ans)

34: Evaluate $\lim_{x\to e} \frac{1-\log_e x}{e-x}$

Soln: $\lim_{x\to e} \frac{1-\log_e x}{e-x} = \lim_{y\to 1} \frac{1-y}{e-e^y}$

Put $\log_e x = y$
$\Rightarrow x = e^y$
As $x\to e, y\to 1$.

$= \lim_{y\to 1}. \frac{(1-y)}{e(1-e^{y-1})} = \frac{1}{e} \lim_{y\to 1} \frac{y-1}{e^{y-1}-1}$

$= \frac{1}{e} \lim_{y-1\to 0} \frac{1}{\left(\frac{e^{y-1}-1}{y-1}\right)} = \frac{1}{e}\times\frac{1}{1} = \frac{1}{e}$ (Ans).

35: Evaluate $\lim_{x\to\infty} \left(1+\frac{2}{3x}\right)^{5x}$

Soln: $\lim_{x\to\infty} \left(1+\frac{2}{3x}\right)^{5x}$

Put $\frac{2}{3x} = y \Rightarrow x = \frac{2}{3y}$
As $x\to\infty, y\to 0$

$= \lim_{y\to 0} (1+y)^{5\times\frac{2}{3y}}$

$$= \lim_{y\to 0}\{(1+y)^{\frac{1}{y}}\}^{\frac{10}{3}} = \{\lim_{y\to 0}(1+y)^{\frac{1}{y}}\}^{\frac{10}{3}} = e^{\frac{10}{3}} \text{ (Ans)}$$

$=$

36: Evaluate $\lim_{x\to 0} \frac{x\log(x+1)}{1-\cos x}$

Soln: $\lim_{x\to 0} \frac{x\log(1+x)}{1-\cos x} = \lim_{x\to 0} \frac{\log(1+x)}{x} \Big/ \left(\frac{1-\cos x}{x^2}\right)$

$$= \lim_{x\to 0} \frac{\log(1+x)}{x} \Big/ \frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{\log(1+x)}{x} \Big/ \frac{2\sin^2\frac{x}{2}}{4\cdot\frac{x^2}{4}}$$

$$= \lim_{x\to 0} \frac{\log(1+x)/x}{\frac{1}{2}\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} = \frac{2\cdot\lim_{x\to 0}\frac{\log(1+x)}{x}}{\lim_{\frac{x}{2}\to 0}\left(\frac{\sin\frac{x}{2}}{x/2}\right)^2} = \frac{2\times 1}{1} = 2.$$

37: Evaluate $\lim_{x\to\infty} \left(\frac{x+2}{x+1}\right)^{x+3}$

Soln: $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3} = \lim_{x\to\infty}\left\{\frac{(x+1)+1}{(x+1)}\right\}^{x+3} = \lim_{x\to\infty}\left\{1+\frac{1}{x+1}\right\}^{x+3}$

$= \lim_{x\to\infty}\{1+f(x)\}^{g(x)}$ where $f(x) = \frac{1}{x+1}$, $g(x) = x+3$

$f(x)\to 0$, as $x\to\infty$, $g(x)\to\infty$, as $x\to\infty$

$$= e^{\lim_{x\to\infty} f(x)\times g(x)} = e^{\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)} = e^{\lim_{x\to\infty}\left(\frac{1+\frac{3}{x}}{1+\frac{1}{x}}\right)}$$

$$= e^{\frac{1+0}{1+0}} = e^1 = e \text{ (Ans)}$$

HOMETASK

① Evaluate: $\lim_{x\to -1}\frac{x^2-1}{x^2+3x+2}$ ② If $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = 80$ then find n.

③ $\lim_{x\to 16}\frac{x^{\frac{3}{4}}-8}{x-16} = ?$ ④ Evaluate: $\lim_{x\to 0}\frac{\sin 3x}{5x}$ (5) $\lim_{x\to 0}\frac{2^{x+3}-8}{x} = ?$

(6) $\lim_{x\to 0}\frac{e^{\sin x}-1}{x} = ?$ (7) $\lim_{x\to 0}(1+3x)^{\frac{2}{x}} = ?$ (8) $\lim_{x\to a}\frac{\sqrt{2a-x}-\sqrt{x}}{a-x} = ?$

(9) If $f(x) = \log x$ then prove that $f(x\cdot y) = f(x)+f(y)$ and $f\left(\frac{x}{y}\right) = f(x)-f(y)$

(10) If $f(x) = \tan x$ then prove that $f(2x) = \frac{2f(x)}{1-[f(x)]^2}$

(11) Evaluate following Limits (i) $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$ (ii) $\lim_{x\to 2}\frac{x^3-x^2-5x+6}{x^2-6x+8}$

(iii) $\lim_{x\to a}\frac{\sqrt[3]{x}-\sqrt[3]{a}}{\sqrt[7]{x}-\sqrt[7]{a}}$ (iv) $\lim_{x\to 1}\frac{\sqrt{x^2-1}+\sqrt{x-1}}{\sqrt{x^2-1}}$ (v) $\lim_{x\to\infty}\frac{2x^2-3x+5}{4x^2-5x+9}$

(vi) $\lim_{x\to\frac{\pi}{4}} \frac{2-\sec^2 x}{1-\tan x}$ (vii) $\lim_{\theta\to 0} \frac{\operatorname{cosec}\theta-\cot\theta}{\theta}$ (viii) $\lim_{x\to 0} \frac{12^x-4^x-3^x+1}{x^2}$

(ix) $\lim_{x\to 0} \frac{e^x-\sin x-1}{x}$ (x) $\lim_{x\to 0}\left(1+\frac{3x}{4}\right)^{\frac{6}{x}}$ (xi) $\lim_{x\to 0}\left(\frac{x+2}{x+1}\right)^x$

(xii) $\lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

(12) Find domain of (i) $f(x)=\frac{x^2-3x+2}{x^2+x-6}$ (ii) $f(x)=\frac{1}{x^3-x}$ (iii) $\frac{1}{\sqrt{x+2}}$ (iv) $\sqrt{2-x}-\frac{1}{\sqrt{9-x^2}}$ (v) $\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(13) Find Range of (i) $f(x)=\frac{x^2}{1+x^2}$ (ii) $f(x)=x^2-6x+7$.

14) Evaluate: $\lim_{x\to 0} \frac{\cos mx-\cos nx}{x^2}$

15) Evaluate: $\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)\tan x$

16) Evaluate: $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-x}$

17) Evaluate: $\lim_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$, $a>b$

18) Evaluate: $\lim_{x\to\frac{1}{2}} \frac{|2x-1|}{2x-1}$

19) Evaluate: $\lim_{x\to\infty} \frac{x}{[x]}$ (Hints: $x-1<[x]\le x$ and Apply Sandwich thm)

20) Evaluate: (a) $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$

(b) $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$

21) Evaluate: $\lim_{x\to\alpha} \frac{x\sin\alpha-\alpha\sin x}{x-\alpha}$ (Hints: put $x-\alpha=y$)

22) Evaluate: $\lim_{x\to 5} \frac{\log_e x-\log_e 5}{x-5}$

23) Evaluate: $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\log_e(1+x)}$

24) Evaluate: $\lim_{x\to 2} \frac{\log_7(2x-3)}{x-2}$

25) Evaluate: $\lim_{x\to 1} \frac{x^2-4x+3}{x^2-6x+5}$

26) Evaluate: $\lim_{x\to 0} \frac{3^x+3^{-x}-2}{x^2}$

27) Evaluate: $\lim_{x\to\infty} \left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)x$

28) Evaluate: $\lim_{x\to 0} \frac{e^{\tan x}-1}{x}$

29) EVALUATE FOLLOWING LIMITS:

(i) $\lim\limits_{x\to\infty} \dfrac{3x^3-4x^2+6x-1}{2x^3+x^2+5x+7}$

(ii) $\lim\limits_{x\to 2} \dfrac{\frac{1}{x^2}-\frac{1}{4}}{x-2}$

(iii) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$

iv) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 2x}$

(v) $\lim\limits_{x\to 1} \dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(vi) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

(vii) $\lim\limits_{x\to \pi} \dfrac{\sin x}{\pi-x}$

(viii) $\lim\limits_{x\to 0} \ln(1+bx)^{\frac{1}{x}}$

(ix) $\lim\limits_{x\to 0} \dfrac{\tan 5x}{\tan 7x}$

(x) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\tan 3x}$

(30) Evaluate following Limits:

(i) $\lim\limits_{x\to 2^+} [x]$ (ii) $\lim\limits_{x\to 3^-} [x]$ (iii) $\lim\limits_{x\to 0} \dfrac{3^x-2^x}{x}$ (iv) $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{c^x-d^x}$

(v) $\lim \dfrac{4^x-5^x}{3^x-2^x}$ (vi) $\lim\limits_{x\to 0} \dfrac{\log(2x+1)}{x}$ (vii) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{3x}$

(vi) $\lim\limits_{x\to 0} \dfrac{\tan ax}{\tan bx}$ (vii) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$ (viii) $\lim\limits_{x\to 0} \dfrac{x}{\tan^{-1}x}$

(ix) $\lim\limits_{x\to 0} \dfrac{1-\cos^3 x}{x\sin 2x}$ (x) $\lim\limits_{x\to 0} \dfrac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}$

(xi) Show that $\lim\limits_{x\to 0} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ doesn't exist.

31) Evaluate $\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{\sin^3 x}$

(32) Find a if (i) $\lim\limits_{x\to \alpha} \dfrac{\tan a(x-\alpha)}{(x-\alpha)} = \dfrac{1}{2}$

(II) $\lim\limits_{x\to 0} \dfrac{e^{ax}-e^x}{x} = 2$

(III) $\lim\limits_{x\to 2} \dfrac{\log_e(2x-3)}{a(x-2)} = 1$

A: DERIVATIVE OF A FUNCTION:

→ If $y = f(x)$ is a function then derivative of y w.r.t x is denoted by $\frac{dy}{dx}$ and defined by $\boxed{\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$ provided that this limit exist.
This Definition of Derivative is called 1st. principle / ab-initio Method.

→ Derivative of $f(x)$ at $x=a$ ($a \in$ dom$f$) is denoted by $\left.\frac{dy}{dx}\right]_{x=a}$ or $f'(a)$ and defined by

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$

→ OTHER NOTATIONS : $D \equiv \frac{d}{dx}$ known as Differential operator.

$$\frac{dy}{dx} = y' = y_1 = Dy = f'(x) = f_1$$

→ The process of finding Derivative of a function is called Differentiation.

→ Note that $\boxed{\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}}$

B: ALGEBRA OF DERIVATIVE:

If u & v are functions of x and 'c' is Any constant then

i. $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

ii. $\frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$

iii. $\frac{d}{dx}(u \times v) = u \times \frac{d}{dx}v + v\frac{d}{dx}u$ (PRODUCT FORMULA)

iv. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$ (QUOTIENT FORMULA)

v. $\frac{d}{dx}(c \cdot u) = c \cdot \frac{d}{dx}(u)$.

VI. $\frac{d}{dx}(c) = 0$

C. DERIVATIVE OF SOME STANDARD FUNCTION:

i. $\frac{d}{dx} x^n = n x^{n-1}$

ii. $\frac{d}{dx} a^x = a^x \ln a$

iii. $\frac{d}{dx} e^x = e^x$

iv. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

v. $\frac{d}{dx} \ln x = \frac{1}{x}$

VI: $\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

VII. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$

$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

EX:1: Find derivative of $\sin x$ using 1st. principle:

Soln: Let $y = f(x) = \sin x$ ——— ①

We have to findout $\frac{dy}{dx}$.

Now $\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$

$= \lim_{h\to 0}\frac{2\cos\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)}{h} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{h}$

$= \lim_{h/2\to 0}\cos\left(x+\frac{h}{2}\right)\times\lim_{h/2\to 0}\frac{\sin(h/2)}{(h/2)}$ As $h\to 0 \Rightarrow h/2\to 0$

$= \cos(x+0)\times 1 = \cos x.$

$\therefore \frac{dy}{dx} = \cos x \Rightarrow \frac{d}{dx}(\sin x) = \cos x$ (Ans).

EX:2: Differentiate $\cos x$ w.r.t x using Definition.

Soln: Let $y = f(x) = \cos x$ ——— ①

We have to find out $\frac{dy}{dx}$

$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

$= \lim_{h\to 0}\frac{-2\sin\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)}{h} = -\lim_{h\to 0}\frac{\sin\left(x+\frac{h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{(h/2)}$

$= -\lim_{h\to 0}\sin(x+h/2)\cdot\lim_{h/2\to 0}\frac{\sin\left(\frac{h}{2}\right)}{(h/2)} = -\sin x\times 1 = -\sin x$

$\therefore \frac{dy}{dx} = -\sin x$ i.e $\frac{d}{dx}\cos x = -\sin x$ (Ans).

Ex: 3: If $y = \tan x$ then find $\frac{dy}{dx}$ using 1st Principle.

Soln: $y = f(x) = \tan x$ ——(1)

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$\lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} = \lim_{h\to 0}\frac{\frac{\sin(x+h)\cdot\cos x - \cos(x+h)\cdot\sin x}{\cos(x+h)\cdot\cos x}}{h}$$

$$= \lim_{h\to 0}\left(\frac{\sin(x+h-x)}{\cos(x+h)\cdot\cos x}\cdot\frac{1}{h}\right) = \lim_{h\to 0}\frac{1}{\cos(x+h)\cdot\cos x}\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \frac{1}{\cos(x+0)\cdot\cos x}\times 1 = \frac{1}{\cos^2 x} = \sec^2 x.$$

$\therefore \frac{dy}{dx} = \sec^2 x$ (Ans.)

Ex4: Find Derivative of x^n w.r.t x using ab-initio Method.

Soln: Let $y = f(x) = x^n$ ——①

We have to find out $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{x+h\to x}\frac{(x+h)^n - x^n}{(x+h)-x} \qquad \text{As } h\to 0,\ x+h\to x$$

$$= n.x^{n-1}$$

$\therefore \frac{dy}{dx} = nx^{n-1}$ i.e $\frac{d}{dx}(x^n) = nx^{n-1}$

Ex-5: Find Derivative of e^x w.r.t x using 1st principle.

Soln: Let $f(x) = e^x$ ——①

We have to find out $f'(x)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x\cdot e^h - e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h} = e^x\times 1 = e^x$$

$\therefore f'(x) = e^x$ (Ans)

Ex-6: Find Derivative of $\log_a x$ w.r.t x.

Let $y = f(x) = \log_a x$ ——— ①

We have to find out $\frac{dy}{dx}$

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log_a(x+h) - \log_a x}{h}$

$= \lim_{h \to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{\log_a\left(1+\frac{h}{x}\right)}{h} = \lim_{h \to 0} \left(\frac{\log_a\left(1+\frac{h}{x}\right)}{\frac{h}{x} \cdot x}\right)$

$= \frac{1}{x} \cdot \lim_{h/x \to 0} \frac{\log_a\left(1+\frac{h}{x}\right)}{(h/x)} = \frac{1}{x} \cdot \frac{1}{\ln a} = \frac{1}{x \ln a}$.

$\therefore \frac{dy}{dx} = \frac{1}{x \ln x}$ i.e $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$ (Ans)

D: DERIVATIVE OF COMPOSITE FUNCTION (CHAIN RULE):

→ If $y = f(u)$ and $u \pm g(x)$ then $\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$

In General $\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dp} \cdots\cdots \frac{dz}{dx}}$

This is called chain Rule.

→ In short, $\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \times g'(x)}$

Ex-7: Find derivative of $2x\sin x - (1+x^2)\sin x$ w.r.t x.

Soln: Let $y = 2x\sin x - (1+x^2)\sin x$

$= [2x - (1+x^2)]\sin x$

$= [2x - x^2 - 1]\sin x$.

We have to find out $\frac{dy}{dx}$

Now $\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\{(2x - x^2 - 1)\sin x\}$

$= (2x - x^2 - 1)\frac{d}{dx}\sin x + \sin x \frac{d}{dx}(2x - x^2 - 1)$

$= (2x - x^2 - 1)\cdot\cos x + \sin x\left\{\frac{d}{dx}(2x) - \frac{d}{dx}(x^2) - \frac{d}{dx}(1)\right\}$

$= (2x - x^2 - 1)\cos x + \sin x\left\{2 \cdot \frac{d}{dx}(x) - 2x - 0\right\}$

$= (2x - x^2 - 1)\cos x + \sin x\{2 \times 1 - 2x\}$

$= (2x - x^2 - 1)\cos x + 2(1 - x)\sin x$.

Ex-8: If $y = \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ then find $\frac{dy}{dx}$.

Soln: $y = \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} = \frac{(\sqrt{a}-\sqrt{x})\frac{d}{dx}(\sqrt{a}+\sqrt{x}) - (\sqrt{a}+\sqrt{x})\frac{d}{dx}(\sqrt{a}-\sqrt{x})}{(\sqrt{a}-\sqrt{x})^2}$$

$$= \frac{(\sqrt{a}-\sqrt{x})\left(0+\frac{1}{2\sqrt{x}}\right) - (\sqrt{a}+\sqrt{x})\left(0-\frac{1}{2\sqrt{x}}\right)}{(\sqrt{a}-\sqrt{x})^2} \qquad \left(\because \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\frac{1}{2\sqrt{x}}\left[(\sqrt{a}-\sqrt{x}) + (\sqrt{a}+\sqrt{x})\right]}{(\sqrt{a}-\sqrt{x})^2} = \frac{\frac{1}{2\sqrt{x}} \times 2\sqrt{a}}{(\sqrt{a}-\sqrt{x})^2} = \frac{\sqrt{a}}{\sqrt{x}(\sqrt{a}-\sqrt{x})^2}$$

Ex-9 If $f(x) = x\tan^{-1}x$ then find $f'(1)$

Soln: $f(x) = x\tan^{-1}x$

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(x\tan^{-1}x) = x\frac{d}{dx}\tan^{-1}x + \tan^{-1}x\frac{d}{dx}(x)$$

$$= x \times \frac{1}{1+x^2} + \tan^{-1}x \times 1 = \frac{x}{1+x^2} + \tan^{-1}x$$

Now $f'(1) = \frac{1}{1+1^2} + \tan^{-1}1 = \frac{1}{2} + \frac{\pi}{4}$ (Ans)

Ex-10 Find $\frac{dy}{dx}$ if $y = \sin x^2$

Soln: $y = \sin x^2 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin x^2) = \cos x^2 \cdot \frac{d}{dx}(x^2)$ by chain Rule

$$= \cos x^2 \times 2x = 2x\cos x^2.$$

Ex-11 Find $\frac{d}{dx}\operatorname{Log}(\tan x)$

Soln: $\frac{d}{dx}\operatorname{Log}(\tan x) = \frac{1}{\tan x} \times \frac{d}{dx}(\tan x) = \frac{1}{\tan x} \cdot \sec^2 x = \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}$

$$= \frac{1}{\sin x \cdot \cos x} = \frac{2}{2\sin x \cdot \cos x} = \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x.$$

Ex-12 If $y = \tan^{-1}(\sec x + \tan x)$ then find $\frac{dy}{dx}$.

Soln: $y = \tan^{-1}(\sec x + \tan x) = \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$

$$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cdot\cos\frac{x}{2}\cdot\sin\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right)$$

$$= \tan^{-1}\left[\frac{(\cos\frac{x}{2} + \sin\frac{x}{2})^2}{(\cos\frac{x}{2} + \sin\frac{x}{2})(\cos\frac{x}{2} - \sin\frac{x}{2})}\right] = \tan^{-1}\left[\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$ Dividing Nr and Dr by $\cos\frac{x}{2}$.

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{x}{2}}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4} + \frac{x}{2}.$$

Now $\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}\left(\frac{x}{2}\right)$

$= 0 + \frac{1}{2}\cdot\frac{d}{dx}(x) = 0+\frac{1}{2}\times 1 = \frac{1}{2}$ (Ans)

13: If $y = \sqrt{\frac{1-\cos x}{1+\cos x}}$ then find $\frac{dy}{dx}$.

Soln: $y = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\frac{1-\cos(2\times\frac{x}{2})}{1+\cos(2\times\frac{x}{2})}} = \sqrt{\frac{2\sin^2(\frac{x}{2})}{2\cos^2(\frac{x}{2})}}$

$= \sqrt{\tan^2(\frac{x}{2})} = \tan(\frac{x}{2})$

Now $\frac{dy}{dx} = \frac{d}{dx}\tan(\frac{x}{2}) = \sec^2(\frac{x}{2})\cdot\frac{d}{dx}(\frac{x}{2})$ by chain Rule

$= \sec^2(\frac{x}{2})\cdot\frac{1}{2}\cdot\frac{d}{dx}(x) = \sec^2(\frac{x}{2})\times\frac{1}{2}\times 1 = \frac{1}{2}\sec^2(\frac{x}{2})$

14/ If $y = \sqrt{1+\sin 2x} + \frac{\sqrt{1+\cos 2x}}{1-\cos 2x}$ then find $\frac{dy}{dx}$.

Soln: $y = \sqrt{1+\sin 2x} + \frac{\sqrt{1+\cos 2x}}{1-\cos 2x}$

$= \sqrt{\sin^2 x+\cos^2 x+2\sin x\cdot\cos x} + \frac{\sqrt{2\cos^2 x}}{2\sin^2 x}$

$= \sqrt{(\sin x+\cos x)^2} + \frac{\sqrt{2}\cos x}{2\sin^2 x} = \sin x+\cos x+\frac{1}{\sqrt{2}}\cdot\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}$

$= \sin x+\cos x+\frac{1}{\sqrt{2}}(\cot x\cdot\operatorname{cosec} x)$

Now $\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\left\{\sin x+\cos x+\frac{1}{\sqrt{2}}\cot x\cdot\operatorname{cosec} x\right\}$

$= \frac{d}{dx}(\sin x)+\frac{d}{dx}(\cos x)+\frac{1}{\sqrt{2}}\frac{d}{dx}(\cot x\cdot\operatorname{cosec} x)$

$= \cos x-\sin x+\frac{1}{\sqrt{2}}\left\{\cot x\cdot\frac{d}{dx}\operatorname{cosec} x+\operatorname{cosec} x\cdot\frac{d}{dx}\cot x\right\}$

$= \cos x-\sin x+\frac{1}{\sqrt{2}}\left\{\cot x\cdot(-\operatorname{cosec} x\cdot\cot x)+\operatorname{cosec} x\cdot(-\operatorname{cosec}^2 x)\right\}$

$= \cos x-\sin x+\frac{1}{\sqrt{2}}\left\{-\cot^2 x\cdot\operatorname{cosec} x-\operatorname{cosec}^3 x\right\}$

$= \cos x-\sin x-\frac{1}{\sqrt{2}}\operatorname{cosec} x(\cot^2 x+\operatorname{cosec}^2 x)$

$= \cos x-\sin x-\frac{\operatorname{cosec} x}{\sqrt{2}}(\cot^2 x+\operatorname{cosec}^2 x)$ (Ans)

15: If $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$ then find $\frac{dy}{dx}$

Soln $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

$= \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$ put $x = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}x$

$\Rightarrow \theta = \frac{1}{2}\cos^{-1}x.$

$= \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan^{-1}\sqrt{\tan^2\theta} = \tan^{-1}(\tan\theta) = \theta = \frac{1}{2}\cos^{-1}x.$

$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x\right) = \frac{1}{2}\cdot\frac{d}{dx}(\cos^{-1}x) = \frac{1}{2}\times -\frac{1}{\sqrt{1-x^2}}$

$= -\frac{1}{2\sqrt{1-x^2}}$ (Ans)

16: If $y = (3x^3+2x^2-x-11)^5$ then find $\frac{dy}{dx}$

Soln: $y = (3x^3+2x^2-x-11)^5$

$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}(3x^3+2x^2-x-11)^5 = 5\cdot(3x^3+2x^2-x-11)^{5-1}\times\frac{d}{dx}(3x^3+2x^2-x-11)$

$= 5(3x^3+2x^2-x-11)^4\times\{3\frac{d}{dx}x^3+2\frac{d}{dx}x^2-\frac{d}{dx}(x)-\frac{d}{dx}(11)\}$

$= 5(3x^3+2x^2-x-11)^4.(3\times3x^2+2\times2x-1-0)$

$= 5(3x^3+2x^2-x-11)^4(9x^2+4x-1)$ (Ans)

17. If $y = \sin x^\circ$ then find $\frac{dy}{dx}$

Soln: $y = \sin x^\circ = \sin(x\times\frac{\pi}{180}) = \sin\left(\frac{\pi x}{180}\right)$

$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \cos\left(\frac{\pi x}{180}\right)\times\frac{d}{dx}\left(\frac{\pi x}{180}\right)$

$= \cos\left(\frac{\pi x}{180}\right)\times\frac{\pi}{180}\cdot\frac{d}{dx}(x) = \cos\left(\frac{\pi x}{180}\right)\cdot\frac{\pi}{180}\cdot 1 = \frac{\pi}{180}\cos x^\circ$ (Ans)

18. If $y = \sin[\cos(\sin x)]$ then find $\frac{dy}{dx}$.

Soln: $y = \sin[\cos(\sin x)]$

$\frac{dy}{dx} = \frac{d}{dx}\sin[\cos(\sin x)] = \cos[\cos(\sin x)]\times\frac{d}{dx}\cos(\sin x)$ (by chain Rule)

$= \cos[\cos(\sin x)].\{-\sin(\sin x)\}.\frac{d}{dx}(\sin x)$ (by chain Rule.)

$= -\cos[\cos(\sin x)].\sin(\sin x)\cdot\cos x$ (Ans)

19. Find derivative of $\log(\sqrt{x-a}+\sqrt{x-b})$ w.r.t x.

Soln; Let $y = \log(\sqrt{x-a}+\sqrt{x-b})$

We have to find out $\frac{dy}{dx}$.

Now $\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\log(\sqrt{x-a}+\sqrt{x-b})$

$= \frac{1}{\sqrt{x-a}+\sqrt{x-b}}\times\frac{d}{dx}(\sqrt{x-a}+\sqrt{x-b})$ by chain Rule.

$= \frac{1}{\sqrt{x-a}+\sqrt{x-b}}\times\left\{\frac{d}{dx}\sqrt{x-a}+\frac{d}{dx}\sqrt{x-b}\right\}$

$$= \frac{1}{\sqrt{x-a}+\sqrt{x-b}} \times \left\{ \frac{1}{2\sqrt{x-a}} \times \frac{d}{dx}(x-a) + \frac{1}{2\sqrt{x-b}} \times \frac{d}{dx}(x-b) \right\}$$

$\left(\because \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}\right.$ And by chain Rule$\left.\right)$

$$= \frac{1}{\sqrt{x-a}+\sqrt{x-b}} \cdot \left\{ \frac{1}{2\sqrt{x-a}} \cdot (1-0) + \frac{1}{2\sqrt{x-b}} (1-0) \right\}$$

$$= \frac{1}{2\{\sqrt{x-a}+\sqrt{x-b}\}} \cdot \frac{(\sqrt{x-b}+\sqrt{x-a})}{\sqrt{x-a} \times \sqrt{x-b}} = \frac{1}{2\sqrt{(x-a)(x-b)}} \text{ (Ans)}$$

20: If $y = \sqrt{x}$ then find $\frac{dy}{dx}$.

Soln. $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}}$$

$$= \frac{1}{2\sqrt{x}} \text{ (Ans)}$$

21. Find $\frac{dy}{dx}$ if (i) $y = \sin^3 x$ (ii) $y = \cos x^4$ (iii) $y = \sqrt[3]{(x^2+1)^2}$ (iv) $y = 2^{\log x}$. (v) $y = \sin(ax+b)$.

Soln:

(i) $y = \sin^3 x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin^3 x) = \frac{d}{dx}(\sin x)^3 = 3(\sin x)^{3-1} \times \frac{d}{dx}(\sin x)$

$= 3\sin^2 x \cdot \cos x$ (Ans)

(ii) $y = \cos x^4 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos x^4 = -\sin x^4 \times \frac{d}{dx} x^4$

$= -\sin x^4 \times 4x^3 = -4x^3 \sin x^4$.

(iii) $y = \sqrt[3]{(x^2+1)^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt[3]{(x^2+1)^2} = \frac{d}{dx}(x^2+1)^{\frac{2}{3}} = \frac{2}{3}(x^2+1)^{\frac{2}{3}-1} \times \frac{d}{dx}(x^2+1)$$

$$= \frac{2}{3} \cdot (x^2+1)^{-\frac{1}{3}} \times (2x+0)$$

$$= \frac{2}{3} \times \frac{1}{(x^2+1)^{1/3}} \times 2x = \frac{4x}{3\sqrt[3]{x^2+1}}$$

(iv) $y = 2^{\log x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx} 2^{\log x} = 2^{\log x} \cdot \ln 2 \cdot \frac{d}{dx}(\log x)$

$$= 2^{\log x} \cdot \ln 2 \times \frac{1}{x} = \frac{2^{\ln x} \cdot \ln 2}{x} \text{ (Ans.)}$$

(v) $y = \sin(ax+b) \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin(ax+b)$

$= \cos(ax+b) \cdot \frac{d}{dx}(ax+b) = \cos(ax+b) \cdot (a \cdot 1 + 0)$

$= a\cos(ax+b)$ (Ans.).

E: DERIVATIVE OF PARAMETRIC FUNCTION:

→ Some time the variable x & y of a function are expressed by function of another variable 't' which is called as a parameter. Such type of representation of a function is called parametric form.

→ For example equation of circle in parametric form can be given by $x = r\cos\theta, y = r\sin\theta$, θ is a parameter.

→ If $\boxed{x = f(t),\ y = g(t)}$ then $\boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}}$

Ex 22: If $\sin x = \frac{2t}{1+t^2}$ and $\tan y = \frac{2t}{1-t^2}$ then find $\frac{dy}{dx}$

Soln: $\sin x = \frac{2t}{1+t^2} \Rightarrow x = \sin^{-1}\left(\frac{2t}{1+t^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$ put $t = \tan\theta \Rightarrow \theta = \tan^{-1}t$

$$\Rightarrow x = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}t.$$

$$\frac{dx}{dt} = \frac{d}{dt}(2\tan^{-1}t) = 2\frac{d}{dt}(\tan^{-1}t) = 2\times\frac{1}{1+t^2} = \frac{2}{1+t^2}$$

Again $\tan y = \frac{2t}{1-t^2} \Rightarrow y = \tan^{-1}\left(\frac{2t}{1-t^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$$\Rightarrow y = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}t.$$

$$\frac{dy}{dt} = \frac{d}{dt}(2\tan^{-1}t) = 2\times\frac{1}{1+t^2} = \frac{2}{1+t^2}.$$

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{2}{1+t^2}\right)}{\left(\frac{2}{1+t^2}\right)} = 1$ (Ans)

F: DERIVATIVE OF A FUNCTION W.R.T ANOTHER FUNCTION:

To differentiate $f(x)$ w.r.t $g(x)$, Let $y = f(x)$, $z = g(x)$ we have to find out $\frac{dy}{dz}$. $\boxed{\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx}}$

Ex-23: Differentiate $\tan x$ w.r.t $\cot x$.

Soln: Let $y = \tan x$ and $z = \cot x$. We have to find out $\frac{dy}{dz}$

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{dz}{dx} = \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\sec^2 x}{-\operatorname{cosec}^2 x} = -\sec^2 x\cdot\sin^2 x$$

$$= -\frac{1}{\cos^2 x}\cdot\sin^2 x = -\frac{\sin^2 x}{\cos^2 x} = -\tan^2 x \text{ (Ans)}$$

G: DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTION:

Ex 24: Differentiate $(\sin x)^2 + x^{\sin x}$ w.r.t x.

Soln: Let $y = (\sin x)^x + (x)^{\sin x}$ ——(1) We have to find out $\frac{dy}{dx}$.

Let $u = (\sin x)^x$ & $v = x^{\sin x}$ then

$$y = u+v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{——(2)}$$

Now $u = (\sin x)^x$

$$\Rightarrow \ln u = \ln (\sin x)^x \Rightarrow \ln u = x \cdot \ln(\sin x)$$

$$\Rightarrow \frac{d}{dx}(\ln u) = \frac{d}{dx}\{x \cdot \ln(\sin x)\}$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{d}{dx} \ln(\sin x) + \ln(\sin x) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \times \frac{1}{\sin x} \times \frac{d}{dx}(\sin x) + \ln(\sin x) \times 1$$

$$= x \times \frac{1}{\sin x} \times \cos x + \ln(\sin x)$$

$$= x \cdot \cot x + \ln(\sin x)$$

$$\Rightarrow \frac{du}{dx} = u[x\cot x + \ln(\sin x)] = (\sin x)^x [x\cot x + \ln(\sin x)] \quad \text{——(3)}$$

Now $v = x^{\sin x} \Rightarrow \ln v = \ln x^{\sin x} = \sin x \cdot \ln x$.

$$\Rightarrow \frac{d}{dx}(\ln v) = \frac{d}{dx}(\sin x \cdot \ln x)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = \sin x \cdot \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(\sin x)$$

$$= \sin x \times \frac{1}{x} + \ln x \times \cos x$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right] \quad \text{——(4)}$$

From (2) $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$$= (\sin x)^x [x\cot x + \ln(\sin x)] + x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]$$

(Ans)

Ex-25: If $y = \log_{\cos x}(\sin x)$ then find $\frac{dy}{dx}$.

Soln: $y = \log_{\cos x}(\sin x)$

$$= \log_e(\sin x) \times \log_{\cos x} e$$

$$= \log_e(\sin x) \times \frac{1}{\log_e(\cos x)}$$

$$\Rightarrow y = \frac{\log_e(\sin x)}{\log_e(\cos x)} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log_e \sin x}{\log_e \cos x}\right)$$

$$= \frac{\log_e \cos x \cdot \frac{d}{dx}\log_e(\sin x) - \log_e \sin x \cdot \frac{d}{dx}\log_e(\cos x)}{[\log_e(\cos x)]^2}$$

$$= \frac{\log_e(\cos x)\cdot \frac{1}{\sin x}\cdot \cos x - \log_e(\sin x)\cdot \frac{1}{\cos x}(-\sin x)}{[\log_e \cos x]^2}$$

$$= \frac{\cot x \cdot \log_e(\cos x) + \tan x \cdot \log_e(\sin x)}{[\log_e(\cos x)]^2}$$

26: If $y = x^{x^x}$ then find $\frac{dy}{dx}$.

Soln: $y = x^{x^x}$ ——— (1)

We have to find out $\frac{dy}{dx}$.

Now $y = x^{x^x} \Rightarrow \log y = \log x^{x^x} = x^x \cdot \log x$.

$\Rightarrow \log(\log y) = \log\{x^x \cdot \log x\} = \log x^x + \log(\log x)$

$\Rightarrow \log(\log y) = x \cdot \log x + \log(\log x)$

$\Rightarrow \frac{d}{dx}\log(\log y) = \frac{d}{dx}\{x \log x + \log(\log x)\} = \frac{d}{dx}(x \cdot \log x) + \frac{d}{dx}\log(\log x)$

$\Rightarrow \frac{d}{dy}\log(\log y) \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}\log x + \log x \cdot \frac{d}{dx}x + \frac{1}{\log x} \times \frac{d}{dx}(\log x)$

$\Rightarrow \frac{1}{\log y} \cdot \frac{d}{dy}(\log y) \cdot \frac{dy}{dx} = x \times \frac{1}{x} + \log x \times 1 + \frac{1}{\log x} \times \frac{1}{x}$

$\Rightarrow \frac{1}{y \log y} \cdot \frac{dy}{dx} = 1 + \log x + \frac{1}{x \log x}$

$\Rightarrow \frac{dy}{dx} = y \log y \left[1 + \log x + \frac{1}{x \log x}\right]$

$= x^{x^x} \cdot \log x^{x^x}\left[1 + \log x + \frac{1}{x \log x}\right]$

$= x^{x^x} \cdot x^x \cdot \log x \left[1 + \log x + \frac{1}{x \log x}\right]$

$= x^{x^x} \cdot x^x \left[\log x + (\log x)^2 + \frac{1}{x}\right]$

$= x^{x^x} \cdot x^x \left[\frac{1}{x} + \log x(1 + \log x)\right]$ (Ans)

H: DERIVATIVE OF IMPLICIT FUNCTION:

→ The function of the form $f(x,y)=0$ in whic y can't be expressed directly in terms of x is called implicit function. Ex→ $x^2+y^2=25$, $x^y=y^x$ etc.

→ To find $\frac{dy}{dx}$, Differentiate both sides of $f(x,y)=0$ w.r.t and apply the formula $\frac{d}{dx}f(y)=\frac{d}{dy}f(y)\cdot\frac{dy}{dx}$.

Ex27: If $x^y-y^x=0$ then find $\frac{dy}{dx}$.

Soln: $x^y-y^x=0 \Rightarrow x^y=y^x$

$\Rightarrow \log x^y = \log y^x \Rightarrow y\log x = x\log y$.

$\Rightarrow \frac{d}{dx}(y\log x) = \frac{d}{dx}(x\log y)$

$\Rightarrow y\cdot\frac{d}{dx}\log x+\log x\frac{d}{dx}(y) = x\cdot\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)$

$\Rightarrow y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx} = x\cdot\frac{d}{dy}\log y\cdot\frac{dy}{dx}+\log y\cdot 1$

$\Rightarrow \frac{y}{x}+\log x\cdot\frac{dy}{dx} = x\times\frac{1}{y}\times\frac{dy}{dx}+\log y$.

$\Rightarrow \log x\cdot\frac{dy}{dx}-\frac{x}{y}\cdot\frac{dy}{dx} = \log y-\frac{y}{x}$

$\Rightarrow \left(\log x-\frac{x}{y}\right)\frac{dy}{dx} = \log y-\frac{y}{x}$

$\Rightarrow \frac{y\log x-x}{y}\cdot\frac{dy}{dx} = \frac{x\log y-y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{x\log y-y}{x}\times\frac{y}{y\log x-x} = \frac{y(x\log y-y)}{x(y\log x-x)}$ (Ans).

HOME TASK

1. Find derivative of following function w.r.t x.

(i) $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$ (ii) $(x-1)(x^2+3)$ (iii) $\frac{(2x+1)(x^2-3)}{x}$.

(iv) $(1+2x^2)\cos x$ (v) $\frac{4^x\cot x}{\sqrt{x}}$

2: Find $\frac{dy}{dx}$ if (i) $y=\operatorname{cosec}(2x^2+3)$ (ii) $y=\tan^3x+\frac{1}{3}\tan x+\frac{?}{3}$

(iii) $y=\sec^2x+\tan^2x$ (iv) $y=\sin^{-1}(2x\sqrt{1-x^2})$

(v) $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (vi) $y=\sin^{-1}(3x-4x^3)$ (vii) $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(viii) $y=\tan^{-1}\left(\frac{1-\sqrt{1-x^2}}{x}\right)$

(ix) $y = x^{\sqrt{x}}$ (x) $y = \log(\log x)$. (xi) $y = \frac{1}{x^4 \sec x}$ (xii) $y = x \sin x$

(xiii) $y = x^2 e^x \sin x$ (xiv) $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ (xv) $y = \cos(1-x^2)^2$

3. If $y = \cos(\sin x^2)$ then find $\frac{dy}{dx}$ at $x = \sqrt{\pi/2}$.

4) Find differential co-efficient of $a^x + \log x \cdot \sin x$.

5) Find $\frac{dy}{dx}$ if (i) $y = \tan^{-1}\left(\frac{ax-b}{bx+a}\right)$ (ii) $y = \tan^{-1}\sqrt{\frac{1+\cos\frac{x}{2}}{1-\cos\frac{x}{2}}}$

(iii) $y = \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$ (iv) $y = \frac{\cot^2 x - 1}{\cot^2 x + 1}$. (v) $y = \sin^n x \cdot \cos nx$

6) Find $f'(x)$ at $x = e$ if $f(x) = \log_x(\log x)$.

7) Find $\frac{dy}{dx}$ if (i) $y = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} - \frac{1}{2}\tan^{-1}x$ (ii) $y = \log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

(iii) $y = \sin^{-1}\sqrt{\sin x}$

(8) If $y = \frac{x}{1+x}, x \neq -1$ then prove that $x^2\frac{dy}{dx} = y^2$

(9) Find derivative of following function:

(i) $y = (\sin x)^{\tan x}$ (ii) $(\cos x)^y = (\sin y)^x$ (iii) $y = (x)^{\sqrt{x}} + (\sqrt{x})^x$

(iv) $y = x^{x^{\sqrt{x}}}$ (v) $x^y = e^{x+y}$

(10) Differentiate following w.r.t x.

(i) $x = at^2, y = 2at,$

(ii) $x = a\sec\theta, y = b\tan\theta$

(iii) $x = 2\cos t - \cos 2t, y = 2\sin t - \sin 2t$.

(11) Find $\frac{dy}{dx}$ if (i) $x^3 + y^3 = 3xy$ (ii) $y = \sin(x+y)$

(iii) $x\sin y + y\sin x = 11$.

→END←