PNS SCHOOL OF ENGINEERING & TECHNOLOGY

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Lecture note on

MECHANICS OF MATERIALS(3rd Semester)

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CENTRE OF GRAVITY

The point, through which the whole weight of the body acts, irrespective of its position, is known as centre of gravity (briefly written as C.G.). It may be noted that everybody has one and only one centre of gravity.

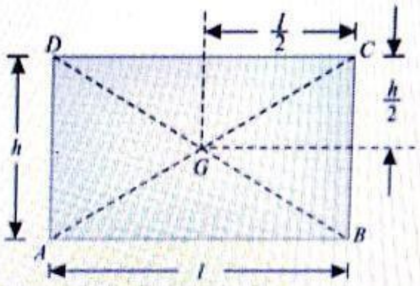
CENTROID

The plane figures (like triangle, quadrilateral, circle etc.) have only areas, but no mass. The centre of area of such figures is known as centroid. The method of finding out the centroid of a figure is the same as that of finding out the centre of gravity of a body.

CENTRE OF GRAVITY BY GEOMETRICAL CONSIDERATIONS:

The centre of gravity of simple figures may be found out from the geometry of the figure as given below.

1. The centre of gravity of uniform rod is at its middle point.
2. The centre of gravity of a rectangle (or a parallelogram) is at the point, where its diagonals meet each other. It is also a middle point of the length as well as the breadth of the rectangle as shown in Fig.



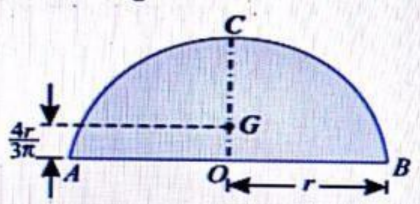
3. The centre of gravity of a triangle is at the point, where the three medians (a median is a line connecting the vertex and middle point of the opposite side) of the triangle meet as shown in Fig.

4. The centre of gravity of a trapezium with parallel sides a and b is at a distance of

$$\frac{h}{3} \times \left(\frac{b + 2a}{b + a} \right)$$

measured form the side b as shown in Fig.

5. **The centre of gravity of a semicircle is at a distance of 4r/3 π from its base measured along the vertical radius as shown in Fig.**

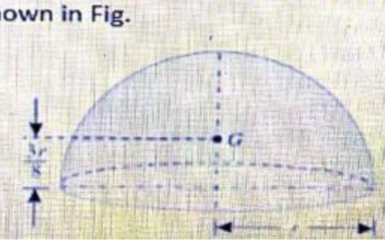


6. The centre of gravity of a circular sector making semi-vertical angle α is at a distance of $\frac{2r}{3}\frac{\sin\alpha}{\alpha}$

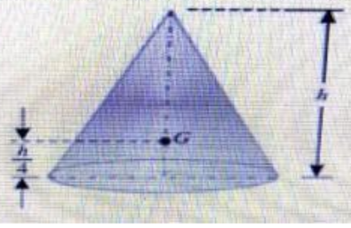
7. The centre of gravity of a cube is at a distance of *l*/ 2 from every face (where *l* is the length of each side).

8. The centre of gravity of a sphere is at a distance of *d* /2 from every point (where *d* is the diameter of the sphere).

9. The centre of gravity of a hemisphere is at a distance of 3r/ 8 from its base, measured along the vertical radius as shown in Fig.



10. **The centre of gravity of right circular solid cone is at a distance of *h* /4 from its base, measured along the vertical axis as shown in Fig.**



AXIS OF REFERENCE

The centre of gravity of a body is always calculated with reference to some assumed axis known as axis of reference (or sometimes with reference to some point of reference). The axis of reference, of plane figures, is generally taken as the lowest line of the figure for calculating $\bar{y}$ and the left line of the figure for calculating $\bar{x}$.

CENTRE OF GRAVITY OF PLANE FIGURES

Let $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of gravity with respect to some axis of reference, then

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots\ldots}{a_1 + a_2 + a_3}$$

and

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + \ldots\ldots}{a_1 + a_2 + a_3 + \ldots}$$

where a_1, a_2, a_3....... etc., are the areas into which the whole figure is divided x_1, x_2, x_3 ..... etc., are the respective co-ordinates of the areas a_1, a_2, a_3 ...... on *X-X* axis with respect to same axis of reference.

y_1, y_2, y_3 ...... etc., are the respective co-ordinates of the areas a_1, a_2, a_3 ...... on *Y-Y* axis with respect to same axis of the reference.

CENTRE OF GRAVITY OF SYMMETRICAL SECTIONS

Sometimes, the given section, whose centre of gravity is required to be found out, is symmetrical about *X-X* axis or *Y-Y* axis. In such cases, the procedure for calculating the centre of gravity of the body is very much simplified, as we have only to calculate either $\bar{x}$ or $\bar{y}$. This is due to the reason that the centre of gravity of the body will lie on the axis of symmetry.

EXAMPLE

Find the centre of gravity of a 100 mm × 150 mm × 30 mm T-section

Solution.

As the section is symmetrical about Y-Y axis, bisecting the web, therefore its centre of gravity will lie on this axis. Split up the section into two rectangles ABCH and DEFG as shown in Fig

Let bottom of the web *FE* be the axis of reference.

(*i*) *Rectangle ABCH*

$$a_1 = 100 \times 30 = 3000 \text{ mm}^2$$

and $$y_1 = \left(150 - \frac{30}{2}\right) = 135 \text{ mm}$$

(*ii*) *Rectangle DEFG*

$$a_2 = 120 \times 30 = 3600 \text{ mm}^2$$

and $$y_2 = \frac{120}{2} = 60 \text{ mm}$$

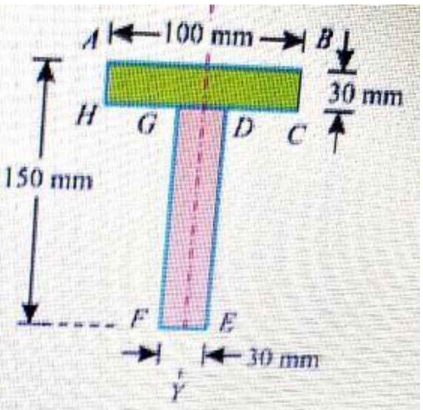


Fig. 6.10.

We know that distance between centre of gravity of the section and bottom of the flange *FE*,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(3000 \times 135) + (3600 \times 60)}{3000 + 3600} \text{ mm}$$

$$= 94.1 \text{ mm} \quad \textbf{Ans.}$$

EXAMPLE

Find the centre of gravity of a channel section 100 mm × 50 mm × 15 mm

Solution.

As the section is symmetrical about X-X axis, therefore its centre of gravity will lie on this axis. Now split up the whole section into three rectangles ABFJ, EGKJ and CDHK as shown in Fig.

Let the face AC be the axis of reference.

(i) *Rectangle ABFJ*

$$a_1 = 50 \times 15 = 750 \text{ mm}^2$$

and $$x_1 = \frac{50}{2} = 25 \text{ mm}$$

(ii) *Rectangle EGKJ*

$$a_2 = (100 - 30) \times 15 = 1050 \text{ mm}^2$$

and $$x_2 = \frac{15}{2} = 7.5 \text{ mm}$$

(iii) *Rectangle CDHK*

$$a_3 = 50 \times 15 = 750 \text{ mm}^2$$

and $$x_3 = \frac{50}{2} = 25 \text{ mm}$$

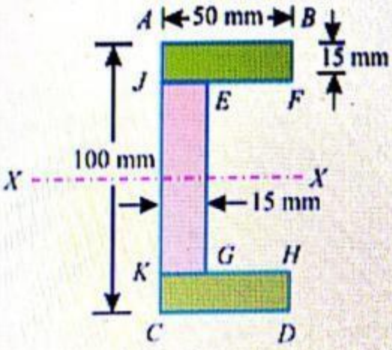


Fig. 6.11.

We know that distance between the centre of gravity of the section and left face of the section AC,

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3}$$

$$= \frac{(750 \times 25) + (1050 \times 7.5) + (750 \times 25)}{750 + 1050 + 750} = 17.8 \text{ mm} \quad \textbf{Ans.}$$

<u>EXAMPLE</u>

An I-section has the following dimensions in mm units : Bottom flange = 300 × 100 Top flange = 150 × 50, Web = 300 × 50. Determine mathematically the position of centre of gravity of the section.

<u>Solution:-</u>

Solution. As the section is symmetrical about Y-Y axis, bisecting the web, therefore its centre of gravity will lie on this axis. Now split up the section into three rectangles as shown in Fig. 6.12.

Let bottom of the bottom flange be the axis of reference.

(i) *Bottom flange*

$$a_1 = 300 \times 100 = 30\,000 \text{ mm}^2$$

and $$y_1 = \frac{100}{2} = 50 \text{ mm}$$

(ii) *Web*

$$a_2 = 300 \times 50 = 15\,000 \text{ mm}^2$$

and $$y_2 = 100 + \frac{300}{2} = 250 \text{ mm}$$

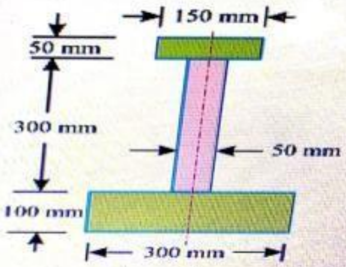


Fig. 6.12.

(iii) *Top flange*

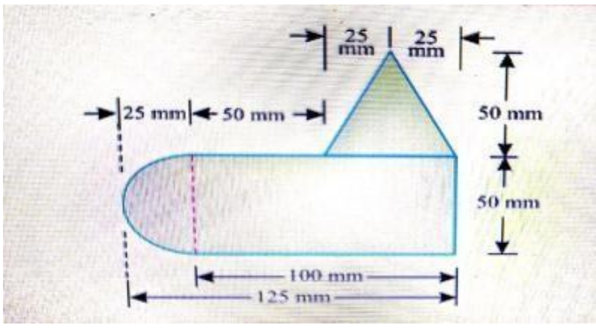
$$a_3 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_3 = 100 + 300 + \frac{50}{2} = 425 \text{ mm}$$

We know that distance between centre of gravity of the section and bottom of the flange,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{(30\,000 \times 50) + (15\,000 \times 250) + (7500 \times 425)}{30\,000 + 15\,000 + 7500} = 160.7 \text{ mm} \quad \textbf{Ans.}$$



Solution. As the section is not symmetrical about any axis, therefore we have to find out the values of both $\bar{x}$ and $\bar{y}$ for the lamina.

Let left edge of circular portion and bottom face rectangular portion be the axes of reference.

(*i*) *Rectangular portion*

$$a_1 = 100 \times 50 = 5000 \text{ mm}^2$$

$$x_1 = 25 + \frac{100}{2} = 75 \text{ mm}$$

and $$y_1 = \frac{50}{2} = 25 \text{ mm}$$

(*ii*) *Semicircular portion*

$$a_2 = \frac{\pi}{2} \times r^2 = \frac{\pi}{2}(25)^2 = 982 \text{ mm}^2$$

$$x_2 = 25 - \frac{4r}{3\pi} = 25 - \frac{4 \times 25}{3\pi} = 14.4 \text{ mm}$$

and $$y_2 = \frac{50}{2} = 25 \text{ mm}$$

(*iii*) *Triangular portion*

$$a_3 = \frac{50 \times 50}{2} = 1250 \text{ mm}^2$$

$$x_3 = 25 + 50 + 25 = 100 \text{ mm}$$

and $$y_3 = 50 + \frac{50}{3} = 66.7 \text{ mm}$$

We know that distance between centre of gravity of the section and left edge of the circular portion,

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = \frac{(5000 \times 75) + (982 \times 14.4) + (1250 \times 100)}{5000 + 982 + 1250}$$

$= 71.1$ mm **Ans.**

Similarly, distance between centre of gravity of the section and bottom face of the rectangular portion,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{(5000 \times 25) + (982 \times 25) + (1250 \times 66.7)}{5000 + 982 + 1250} \text{ mm}$$

$= 32.2$ mm **Ans.**

<u>MOMENT OF INERTIA</u>

The moment of a force (P) about a point, is the product of the force and perpendicular distance (x) between the point and the line of action of the force (i.e. P.x). This moment is also called first moment of force. If this moment is again multiplied by the perpendicular distance (x) between the point and the line of action of the force i.e. P.x (x) = Px^2, then this quantity is called moment of the moment of a force or second moment of force or moment of inertia (briefly written as M.I.)

<u>MOMENT OF INERTIA OF A PLANE AREA</u>

Consider a plane area, whose moment of inertia is required to be found out. Split up the whole area into a number of small elements.

Let $a_1, a_2, a_3, \ldots$ = Areas of small elements, and

$r_1, r_2, r_3, \ldots$ = Corresponding distances of the elements from the line about which the moment of inertia is required to be found out.

Now the moment of inertia of the area,

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + \ldots$$
$$= \Sigma a r^2$$

UNITS OF MOMENT OF INERTIA

As a matter of fact the units of moment of inertia of a plane area depend upon the units of the area and the length. e.g.

1. If area is in m^2 and the length is also in m, the moment of inertia is expressed in m^4
2. If area in mm^2 and the length is also in mm, then moment of inertia is expressed in mm^4

MOMENT OF INERTIA BY INTEGRATION

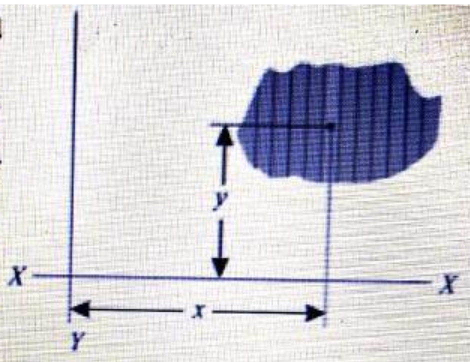
The moment of inertia of an area may also be found out by the method of integration as discussed below:

Consider a plane figure, whose moment of inertia is required to be found out about X-X axis and Y-Y axis as shown in Fig. Let us divide the whole area into a no. of strips. Consider one of these strips.

Let dA = Area of the strip

x = Distance of the centre of gravity of the strip on X-X axis and

y = Distance of the centre of gravity of the strip on Y-Y axis.



Moment of inertia by integration.

We know that the moment of inertia of the strip about Y-Y axis

$= dA \cdot x^2$

Now the moment of inertia of the whole area may be found out by integrating above equation. i.e.,

$$I_{YY} = \Sigma\, dA \cdot x^2$$

Similarly $$I_{XX} = \Sigma\, dA \cdot y^2$$

MOMENT OF INERTIA OF A RECTANGULAR SECTION:

Consider a rectangular section ABCD as shown in Fig. whose moment of inertia is required to be found out.

Let b = Width of the section and

d = Depth of the section.

Now consider a strip PQ of thickness dy parallel to X-X axis and at a distance y from it as shown in the figure

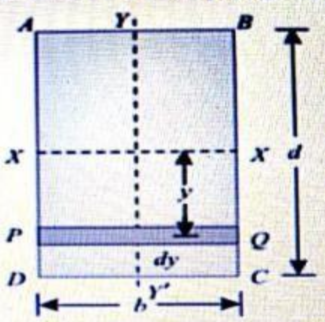
∴ Area of the strip

$= b.dy$

We know that moment of inertia of the strip about X-X axis,

$= \text{Area} \times y^2 = (b.\ dy)\ y^2 = b.\ y^2.\ dy$

Now *moment of inertia of the whole section may be found out by integrating the above equation for the whole length of the lamina i.e. from $-\frac{d}{2}$ to $+\frac{d}{2}$.



Rectangular section

$$I_{xx} = \int_{-\frac{d}{2}}^{+\frac{d}{2}} b.y^2.dy = b\int_{-\frac{d}{2}}^{+\frac{d}{2}} y^2\ .\ dy$$

$$= b\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}} = b\left[\frac{(d/2)^3}{3} - \frac{(-d/2)^3}{3}\right] = \frac{bd^3}{12}$$

Similarly, $$I_{YY} = \frac{db^3}{12}$$

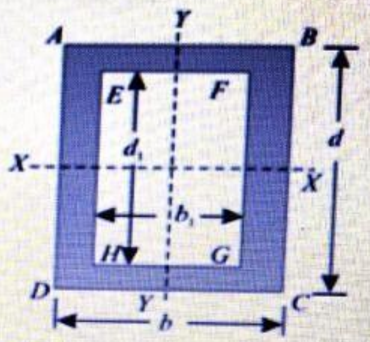
MOMENT OF INERTIA OF A HOLLOW RECTANGULAR SECTION:

Consider a hollow rectangular section, in which *ABCD* is the main section and *EFGH* is the cut out section as shown in Fig

Let b = Breadth of the outer rectangle.

d = Depth of the outer rectangle and

b_1, d_1 = Corresponding values for the cut out rectangle.



Hollow rectangular section

We know that the moment of inertia, of the outer rectangle *ABCD* about *X-X* axis

$$= \frac{bd^3}{12} \quad ...(i)$$

and moment of inertia of the cut out rectangle *EFGH* about *X-X* axis

$$= \frac{b_1 d_1^3}{12} \quad ...(ii)$$

∴ M.I. of the hollow rectangular section about *X-X* axis.

I_{XX} = M.I. of rectangle *ABCD* – M.I. of rectangle *EFGH*

$$= \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$$

Similarly,
$$I_{yy} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12}$$

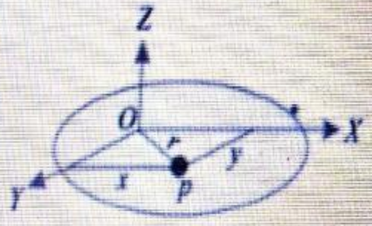
THEOREM OF PERPENDICULAR AXIS:

It states, If I_{XX} and I_{YY} be the moments of inertia of a plane section about two perpendicular axis meeting at O, the moment of inertia I_{ZZ} about the axis Z-Z perpendicular to the plane and passing through the intersection of X-X and Y-Y is given by:

$$I_{ZZ} = I_{XX} + I_{YY}$$

Proof :

Consider a small lamina (*P*) of area *da* having co-ordinates as *x* and *y* along *OX* and *OY* two mutually perpendicular axes on a plane section as shown in Fig.



Theorem of perpendicular axis

Now consider a plane *OZ* perpendicular to *OX* and *OY*. Let (*r*) be the distance of the lamina (*P*) from *Z-Z* axis such that *OP* = *r*.

From the geometry of the figure, we find that

$$r^2 = x^2 + y^2$$

We know that the moment of inertia of the lamina *P* about *X-X* axis,

$$I_{XX} = da \cdot y^2 \qquad [\because I = \text{Area} \times (\text{Distance})^2]$$

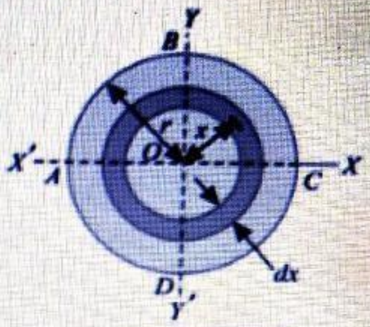
Similarly,
$$I_{YY} = da \cdot x^2$$

and
$$I_{ZZ} = da \cdot r^2 = da\,(x^2 + y^2) \qquad (\because r^2 = x^2 + y^2)$$

$$= da \cdot x^2 + da \cdot y^2 = I_{YY} + I_{XX}$$

MOMENT OF INERTIA OF A CIRCULAR SECTION:

Consider a circle *ABCD* of radius (*r*) with centre *O* and *X-X'* and *Y-Y'* be two axes of reference through *O* as shown in Fig.



Circular section.

Now consider an elementary ring of radius *x* and thickness *dx*. Therefore area of the ring.

$$da = 2\pi x . dx$$

and moment of inertia of ring, about *X-X* axis or *Y-Y* axis

$$= \text{Area} \times (\text{Distance})^2$$

$$= 2\pi x . dx \times x^2$$

$$= 2\pi x^3 . dx$$

Now moment of inertia of the whole section, about the central axis, can be found out by integrating the above equation for the whole radius of the circle *i.e.*, from 0 to *r*.

$$\therefore \qquad I_{ZZ} = \int_0^r 2\pi x^3 . dx = 2\pi \int_0^r x^3 . dx$$

$$I_{ZZ} = 2\pi \left[\frac{x^4}{4}\right]_0^r = \frac{\pi}{2}(r)^4 = \frac{\pi}{32}(d)^4 \qquad \left(\text{substituting } r = \frac{d}{2}\right)$$

We know from the Theorem of Perpendicular Axis that

$$I_{XX} + I_{YY} = I_{ZZ}$$

$$\therefore I_{XX} = I_{YY} = \frac{I_{ZZ}}{2} = \frac{1}{2} \times \frac{\pi}{32}(d)^4 = \frac{\pi}{64}(d)^4$$

MOMENT OF INERTIA OF A HOLLOW CIRCULAR SECTION:

Consider a hollow circular section as shown in Fig. whose moment of inertia is required to be found out.

It states, *If the moment of inertia of a plane area about an axis through its centre of gravity is denoted by I_G, then moment of inertia of the area about any other axis AB, parallel to the first, and at a distance h from the centre of gravity is given by:*

$$I_{AB} = I_G + ah^2$$

where I_{AB} = Moment of inertia of the area about an axis *AB*,

I_G = Moment of Inertia of the area about its centre of gravity

a = Area of the section, and

and moment of inertia of the whole section about an axis passing through its centre of gravity,

$$I_G = \sum \delta a . y^2$$

$\therefore$ Moment of inertia of the section about the axis *AB*,

$$I_{AB} = \sum \delta a (h + y)^2 = \sum \delta a (h^2 + y^2 + 2hy)$$

THEOREM OF PARALLEL AXIS:

$$= (\sum h^2 . \delta a) + (\sum y^2 . \delta a) + (\sum 2hy . \delta a)$$

$$= ah^2 + I_G + 0$$

It may be noted that $\sum h^2 . \delta a = ah^2$ and $\sum y^2 . \delta a = I_G$ [as per equation (*i*) above] and $\sum \delta a . y$ is the algebraic sum of moments of all the areas, about an axis through centre of gravity of the section and is equal to $a . \bar{y}$, where $\bar{y}$ is the distance between the section and the axis passing through the centre of gravity, which obviously is zero

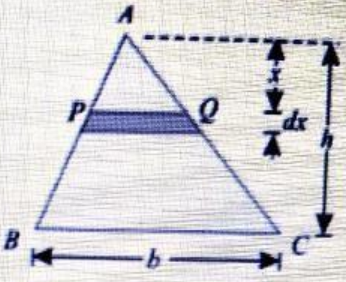
MOMENT OF INERTIA OF A TRIANGULAR SECTION:

Consider a triangular section ABC whose moment of inertia is required to be found out.

Let b = Base of the triangular section and

h = Height of the triangular section.

Now consider a small strip PQ of thickness dx at a distance of x from the vertex A as shown in Fig. From the geometry of the figure, we find that the two triangles APQ and ABC are similar. Therefore



Triangular section

$$\frac{PQ}{BC} = \frac{x}{h} \quad \text{or} \quad PQ = \frac{BC.x}{h} = \frac{bx}{h} \qquad (\because BC = \text{base} = b)$$

We know that area of the strip PQ

$$= \frac{bx}{h} \cdot dx$$

and moment of inertia of the strip about the base BC

$$= \text{Area} \times (\text{Distance})^2 = \frac{bx}{h} dx\,(h - x)^2 = \frac{bx}{h}(h - x)^2 dx$$

Now moment of inertia of the whole triangular section may be found out by integrating the above equation for the whole height of the triangle *i.e.*, from 0 to h

$$I_{BC} = \int_0^h \frac{bx}{h}(h - x)^2\,dx$$

$$= \frac{b}{h}\int_0^h x\,(h^2 + x^2 - 2hx)\,dx$$

$$= \frac{b}{h}\int_0^h (xh^2 + x^3 - 2hx^2)\,dx$$

$$= \frac{b}{h}\left[\frac{x^2h^2}{2} + \frac{x^4}{4} - \frac{2hx^3}{3}\right]_0^h = \frac{bh^3}{12}$$

We know that distance between centre of gravity of the triangular section and base BC.

$$d = \frac{h}{3}$$

$\therefore$ Moment of inertia of the triangular section about an axis through its centre of gravity and parallel to X-X axis,

$$I_G = I_{BC} - ad^2 \qquad (\because I_{XX} = I_G + a\,h^2)$$

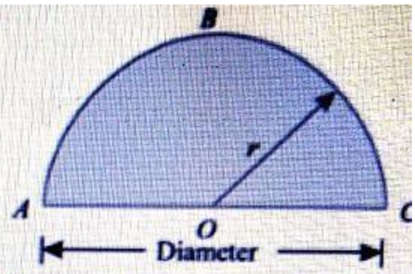
$$= \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

MOMENT OF INERTIA OF A SEMICIRCULAR SECTION:

Consider a semicircular section *ABC* whose moment of inertia is required to be found out as shown in Fig. ....

Let r = Radius of the semicircle.

We know that moment of inertia of the semicircular section about the base *AC* is equal to half the moment of inertia of the circular section about *AC*. Therefore moment of inertia of the semicircular section *ABC* about the base *AC*.



Semicircular section *ABC*.

$$I_{AC} = \frac{1}{2} \times \frac{\pi}{64} \times (d)^4 = 0.393\, r^4$$

We also know that area of semicircular section.

$$a = \frac{1}{2} \times \pi r^2 \; \frac{\pi r^2}{2}$$

and distance between centre of gravity of the section and the base *AC*.

$$h = \frac{4r}{3\pi}$$

∴ Moment of inertia of the section through its centre of gravity and parallel to x-x axis.

$$I_G = I_{AC} - ah^2 = \left[\frac{\pi}{8} \times (r)^4\right] - \left[\frac{\pi r^2}{2}\left(\frac{4r}{3\pi}\right)^2\right]$$

$$= \left[\frac{\pi}{8} \times (r)^4\right] - \left[\frac{8}{9\pi} \times (r)^4\right] = 0.11\, r^4$$

Note. The moment of inertia about y-y axis will be the same as that about the base *AC* *i.e.*, $0.393\, r^4$

MOMENT OF INERTIA OF A COMPOSITE SECTION

The moment of inertia of a composite section may be found out by the following steps :

1. First of all, split up the given section into plane areas (*i.e.*, rectangular, triangular, circular etc., and find the centre of gravity of the section).
2. Find the moments of inertia of these areas about their respective centres of gravity.
3. Now transfer these moment of inertia about the required axis (*AB*) by the Theorem of Parallel Axis, *i.e.*,

$$I_{AB} = I_G + ah^2$$

where I_G = Moment of inertia of a section about its centre of gravity and parallel to the axis.

a = Area of the section.

h = Distance between the required axis and centre of gravity of the section.

4. The moments of inertia of the given section may now be obtained by the algebraic sum of the moment of inertia about the required axis.

EXAMPLE

Find the moment of inertia of a rectangular section 30 mm wide and 40 mm deep about

X-X axis and Y-Y axis.

Solution:-

Given: Width of the section (b) = 30 mm and
Depth of the section (d) = 40 mm.

We know that moment of inertia of the section about an axis passing through its centre of gravity and parallel to X-X axis,

$$I_{XX} = \frac{bd^3}{12} = \frac{30 \times (40)^3}{12} = 160 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

Similarly $$I_{YY} = \frac{db^3}{12} = \frac{40 \times (30)^3}{12} = 90 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

EXAMPLE

Find the moment of inertia of a hollow rectangular section about its centre of gravity if the external dimensions are breadth 60 mm, depth 80 mm and internal dimensions are breadth 30 mm and depth 40 mm respectively.

Solution. Given: External breadth (b) = 60 mm; External depth (d) = 80 mm ; Internal breadth (b_1) = 30 mm and internal depth (d_1) = 40 mm.

We know that moment of inertia of hollow rectangular section about an axis passing through its centre of gravity and parallel to X-X axis,

$$I_{XX} = \frac{bd^3}{12} - \frac{b_1 d_1^3}{12} = \frac{60\ (80)^3}{12} - \frac{30\ (40)^3}{12} = 2400 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

Similarly, $$I_{YY} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12} = \frac{80\ (60)^3}{12} - \frac{40\ (30)^3}{12} = 1350 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

EXAMPLE

Find the moment of inertia of a circular section of 50 mm diameter about an axis passing through its centre.

Solution.

Given: Diameter (d) = 50 mm We know that moment of inertia of the circular section about an axis passing through its centre.

$$I_{XX} = \frac{\pi}{64}(d)^4 = \frac{\pi}{64} \times (50)^4 = 307 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

EXAMPLE

Find the moment of inertia of a T-section with flange as 150 mm × 50 mm and web as 150 mm × 50 mm about X-X and Y-Y axes through the centre of gravity of the section.

First of all, let us find out centre of gravity of the section.

As the section is symmetrical about Y-Y axis, therefore its centre of gravity will lie on this axis. Split up the whole section into two rectangles *viz.*, 1 and 2 as shown in figure. Let bottom of the web be the axis of reference.

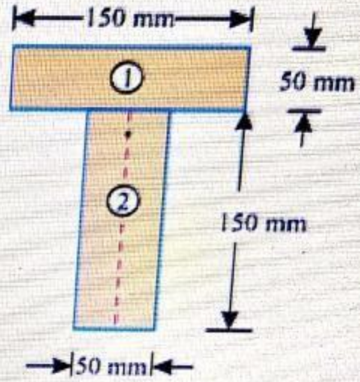


Fig. 7.14.

(*i*) *Rectangle* (1)

$$a_1 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_1 = 150 + \frac{50}{2} = 175 \text{ mm}$$

(*ii*) *Rectangle* (2)

$$a_2 = 150 \times 50 = 7500 \text{ mm}^2$$

and $$y_2 = \frac{150}{2} = 75 \text{ mm}$$

We know that distance between centre of gravity of the section and bottom of the web,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(7500 \times 175) + (7500 \times 75)}{7500 + 7500} = 125 \text{ mm}$$

Moment of inertia about X-X axis

We also know that M.I. of rectangle (1) about an axis through its centre of gravity and parallel to X-X axis.

$$I_{G1} = \frac{150\,(50)^3}{12} = 1.5625 \times 10^6 \text{ mm}^4$$

and distance between centre of gravity of rectangle (1) and X-X axis,

$$h_1 = 175 - 125 = 50 \text{ mm}$$

∴ **Moment of inertia of rectangle (1) about X-X axis**

$$I_{G1} + a_1 h_1^2 = (1.5625 \times 10^6) + [7500 \times (50)^2] = 20.3125 \times 10^6 \text{ mm}^4$$

Similarly, moment of inertia of rectangle (2) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G2} = \frac{50\,(150)^3}{12} = 14.0625 \times 10^6 \text{ mm}^4$$

and distance between centre of gravity of rectangle (2) and X-X axis,

$$h_2 = 125 - 75 = 50 \text{ mm}$$

∴ Moment of inertia of rectangle (2) about X-X axis

$$= I_{G2} + a_2 h_2^2 = (14.0625 \times 10^6) + [7500 \times (50)^2] = 32.8125 \times 10^6 \text{ mm}^4$$

Now moment of inertia of the whole section about X-X axis,

$$I_{XX} = (20.3125 \times 10^6) + (32.8125 \times 10^6) = 53.125 \times 10^6 \text{ mm}^4 \quad \textbf{Ans.}$$

Moment of inertia about Y-Y axis

We know that M.I. of rectangle (1) about Y-Y axis

$$= \frac{50\,(150)^3}{12} = 14.0625 \times 10^6 \text{ mm}^4$$

and moment of inertia of rectangle (2) about Y-Y axis,

$$= \frac{150\,(50)^3}{12} = 1.5625 \times 10^6 \text{ mm}^4$$

Now moment of inertia of the whole section about Y-Y axis,

$$I_{YY} = (14.0625 \times 10^6) + (1.5625 \times 10^6) = 15.625 \times 10^6 \text{ mm}^4 \quad \textbf{Ans.}$$

<u>EXAMPLE</u>

An I-section is made up of three rectangles as shown in Fig. Find the moment of inertia of the section about the horizontal axis passing through the centre of gravity of the section.

Solution. First of all, let us find out centre of gravity of the section. As the section is symmetrical about Y-Y axis, therefore its centre of gravity will lie on this axis.

Split up the whole section into three rectangles 1, 2 and 3 as shown in Fig. 7.15. Let bottom face of the bottom flange be the axis of reference.

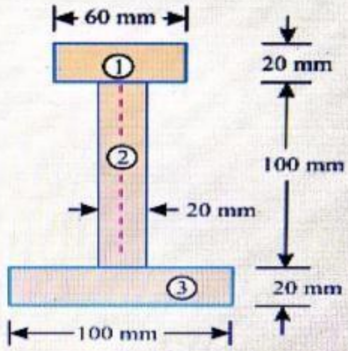


Fig. 7.15.

(*i*) *Rectangle* 1

$$a_1 = 60 \times 20 = 1200 \text{ mm}$$

and $$y_1 = 20 + 100 + \frac{20}{2} = 130 \text{ mm}$$

(*ii*) *Rectangle* 2

$$a_2 = 100 \times 20 = 2000 \text{ mm}^2$$

and $$y_2 = 20 + \frac{100}{2} = 70 \text{ mm}$$

(*iii*) *Rectangle* 3

$$a_3 = 100 \times 20 = 2000 \text{ mm}^2$$

and $$y_3 = \frac{20}{2} = 10 \text{ mm}$$

We know that the distance between centre of gravity of the section and bottom face,

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = \frac{(1200 \times 130) + (2000 \times 70) + (2000 \times 10)}{1200 + 2000 + 2000} \text{ mm}$$

$$= 60.8 \text{ mm}$$

We know that moment of inertia of rectangle (1) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G1} = \frac{60 \times (20)^3}{12} = 40 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (1) and X-X axis,

$$h_1 = 130 - 60.8 = 69.2 \text{ mm}$$

∴ Moment of inertia of rectangle (1) about X-X axis,

$$= I_{G1} + a_1 h_1^2 = (40 \times 10^3) + [1200 \times (69.2)^2] = 5786 \times 10^3 \text{ mm}^4$$

Similarly, moment of inertia of rectangle (2) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G2} = \frac{20 \times (100)^3}{12} = 1666.7 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (2) and X-X axis,

$$h_2 = 70 - 60.8 = 9.2 \text{ mm}$$

∴ Moment of inertia of rectangle (2) about X-X axis,

$$= I_{G2} + a_2 h_2^2 = (1666.7 \times 10^3) + [2000 \times (9.2)^2] = 1836 \times 10^3 \text{ mm}^4$$

Now moment of inertia of rectangle (3) about an axis through its centre of gravity and parallel to X-X axis,

$$I_{G3} = \frac{100 \times (20)^3}{12} = 66.7 \times 10^3 \text{ mm}^4$$

and distance between centre of gravity of rectangle (3) and X-X axis,

$$h_3 = 60.8 - 10 = 50.8 \text{ mm}$$

∴ Moment of inertia of rectangle (3) about X-X axis,

$$= I_{G3} + a_3 h_3^2 = (66.7 \times 10^3) + [2000 \times (50.8)^2] = 5228 \times 10^3 \text{ mm}^4$$

Now moment of inertia of the whole section about X-X axis,

$$I_{XX} = (5786 \times 10^3) + (1836 \times 10^3) + (5228 \times 10^3) = 12\,850 \times 10^3 \text{ mm}^4$$ **Ans.**

Simple Stresses and strains

Stress:

Stress of a material is defined as the resistance offered by the material per unit cross sectional area of the member subjected to external force.

Mathematically, stress is expressed as

$$Stress = \frac{\text{Resistance}}{\text{Cross sectional area}} = \frac{Force}{C/S\,area}$$

$$\sigma = \frac{R}{A} = \frac{F}{A}$$

Where, F = externally applied force, and

A = Cross sectional area of the member

Unit of stress in S.I system of units is N/mm^2.

Depending upon the nature of the force, stresses are of various types. Force acting on a structural member may be ***perpendicular*** to the plane or ***tangential*** to the plane. The force may tend to bend the plane about any of the axis on the plane or it may twist the plane about the axis perpendicular to the plane.

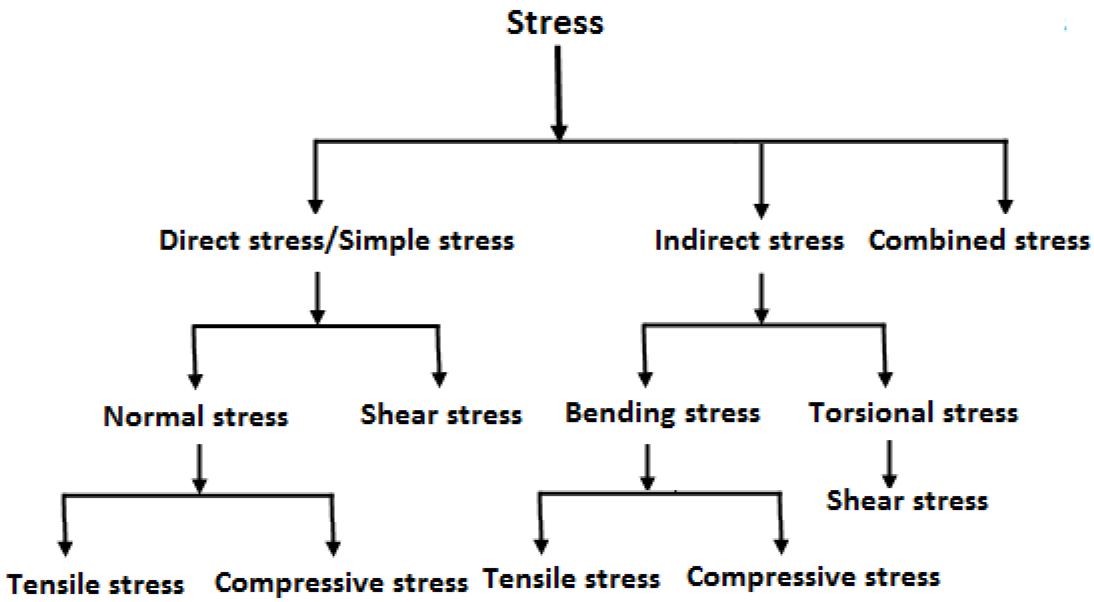


Fig. Stress Types

Strain: *Strain* is the measure of deformation produced in the structural member by the applied load. It is expressed as the ratio of change in dimension to the original dimension of the member before application of load. It is denoted by 'ε' (Greek small letter epsilon). Strain is a dimensionless quantity as it is the ratio of same units.

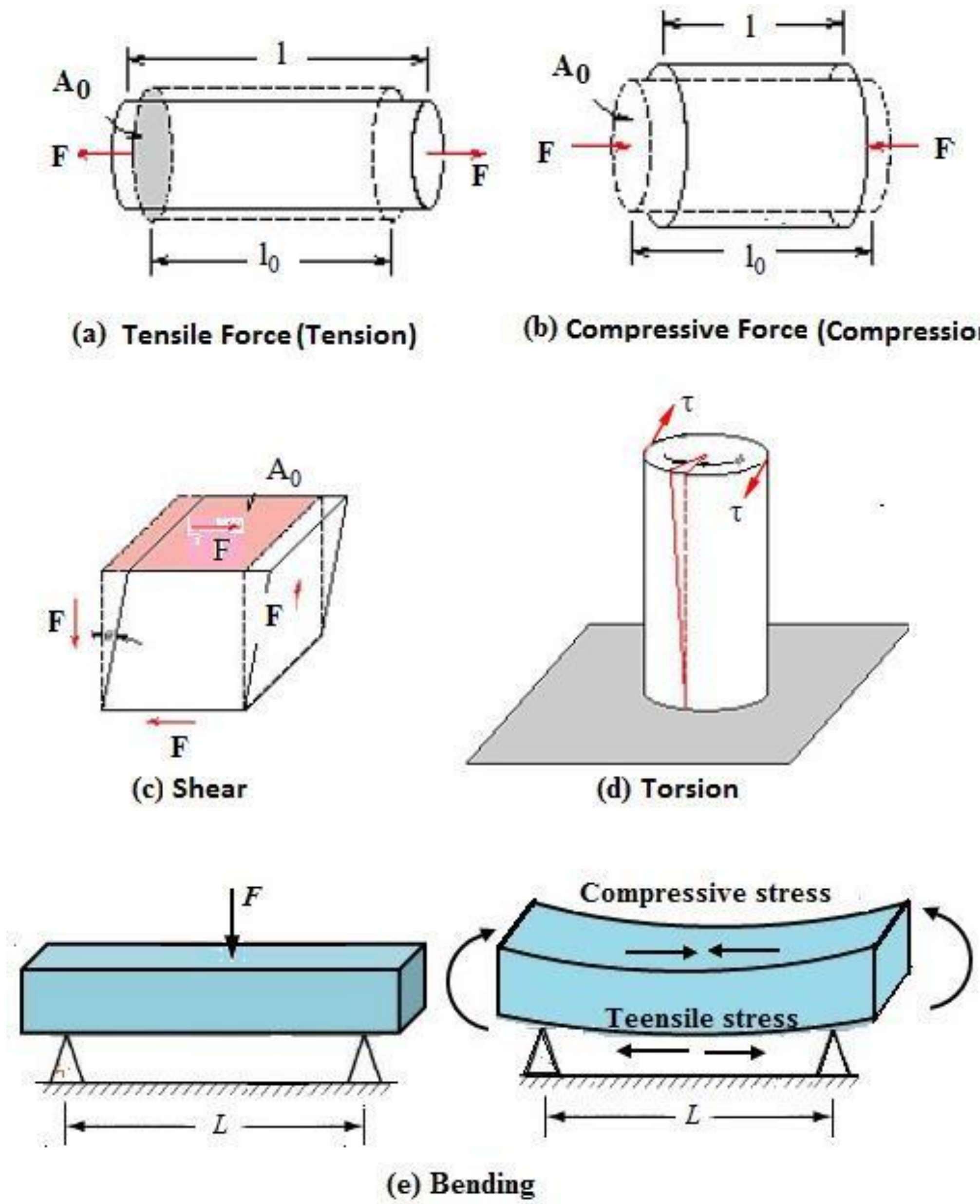


Figure. Types of Loading

The dimension may be length, area, volume or the change in angle. The nomenclature of the strain follows according to the dimension.

Direct Stress: Direct stress also called *simple stress* is induced due to direct loading condition on the plane. It is of two types.

a. Normal stress
b. Tangential stress or shear stress

Normal stress: Stress developed on a plane subjected to axial loading (load perpendicular to the plane) is called *normal stress*. Normal stress is of two types.

a. Tensile stress
b. Compressive stress

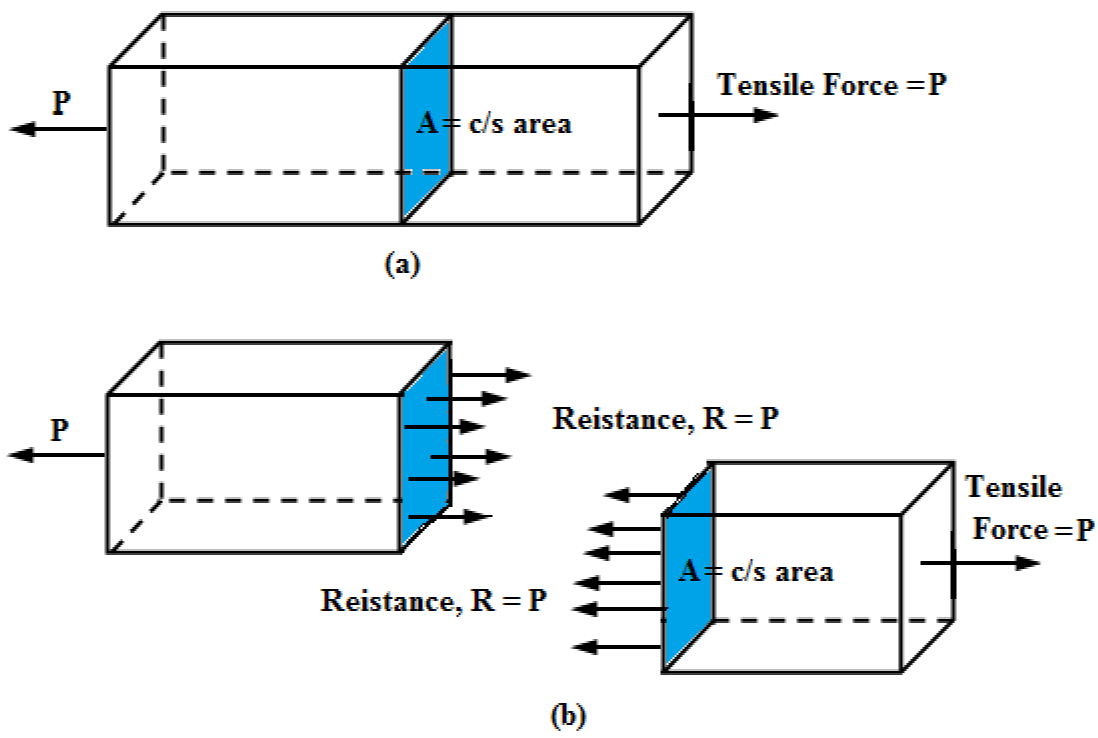


Fig. Tensile stress

Tensile stress: The stress induced on a plane of a body subjected to axial tensile load (two equal and opposite pulls) is known as *tensile stress*.

When a body of uniform cross section e.g., a steel rod is subjected to an axial pull or tensile force, it has a tendency to elongate. The stress induced at any cross section of the rod is known as ***tensile stress***.

It is the resistance of the material l of the member per unit area of cross section normal to the load.

$$Tenile\ stress = \frac{\text{Resistance}}{\text{Original cross sectional area}} = \frac{Tensile\ Force}{C/S\ area}$$

$$\sigma = \frac{R}{A_0} = \frac{F}{A_0}$$

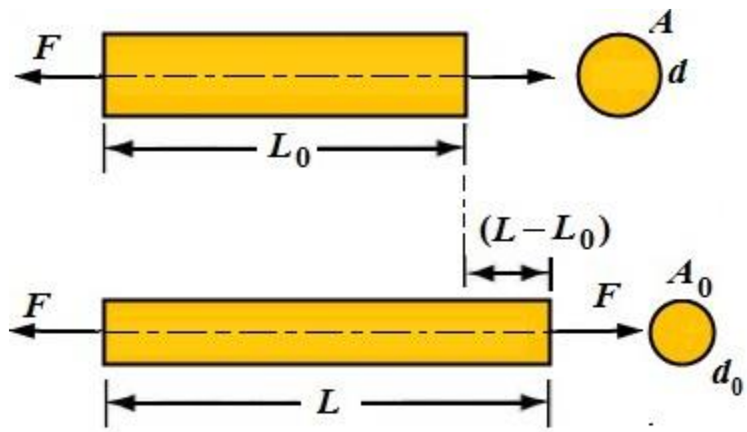


Fig. Axial Tensile Load

Tensile strain: It is the ration of increase in length to the original length of the member subjected to axial tensile force.

$$\varepsilon_l = \frac{\delta L}{L_0}$$

Where, L_0 = Original length of the member before being subjected to load

L = Final length due to the applied load

$\delta L = L - L_0$ = increase in length

Compressive stress: When a structural member is subjected to compressive force, it has a tendency to decrease in length. The stress induced in the member by virtue of the resistance offered by the material of the member is known as *compressive stress*.

The stress induced in the material of the member by virtue of the resistance offered by it against decrease in length due to axial compressive load is known as compressive stress.

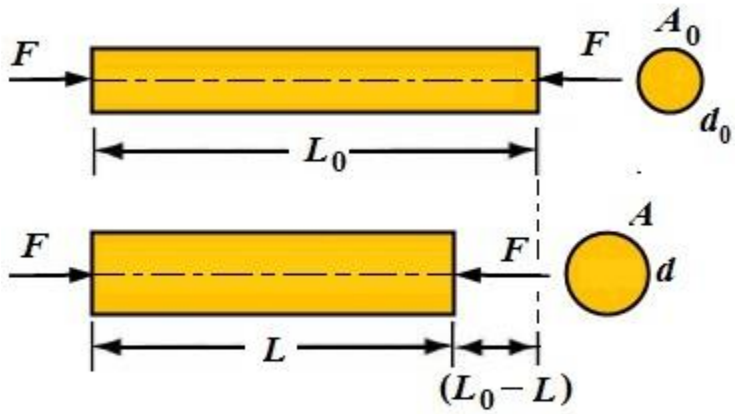


Fig. Axial Compressive Load

$$Comprssive\ stress = \frac{\text{Resistance}}{\text{Original cross sectional area}} = \frac{Compressive\ Force}{C/S\ area}$$

$$\sigma = \frac{R}{A_0} = \frac{F}{A_0}$$

Compressive strain: It is the ration of decrease in length to the original length of the member subjected to axial compressive force.

Lateral strain: It is the ratio of change in lateral dimension to the original lateral dimension. Strain

$$Lateral\ strain, \varepsilon_d = \frac{\text{Change in lateral dimension}}{\text{Original lateral dimension}} = \frac{\delta d}{d} = \frac{d - d_0}{d_0}$$

Strains in the direction transverse to the direction of load are called ***lateral strains***. Lateral strains have a nature/sense opposite to that of the linear or primary strain. Lateral strains are also called ***transverse strains*** or ***secondary strains***.

Elongation: Increase in the dimension, length, breath or depth (height) of the structural member on account of externally applied load is known as elongation.

Contraction: Decrease in the dimension of structural member on account of externally applied load is known as contraction.

Poisson's ratio: The ratio of lateral strain to the linear strain is known as ***Poisson's ratio***. It is denoted by 'ν' (Greek small letter Nu). Within elastic limit, in most material, this ratio remains fairly constant.

$$Poisson's\ ratio, \nu = \frac{Lateral\ strain}{Linear\ strain} = \frac{\varepsilon_d}{\varepsilon_l}$$

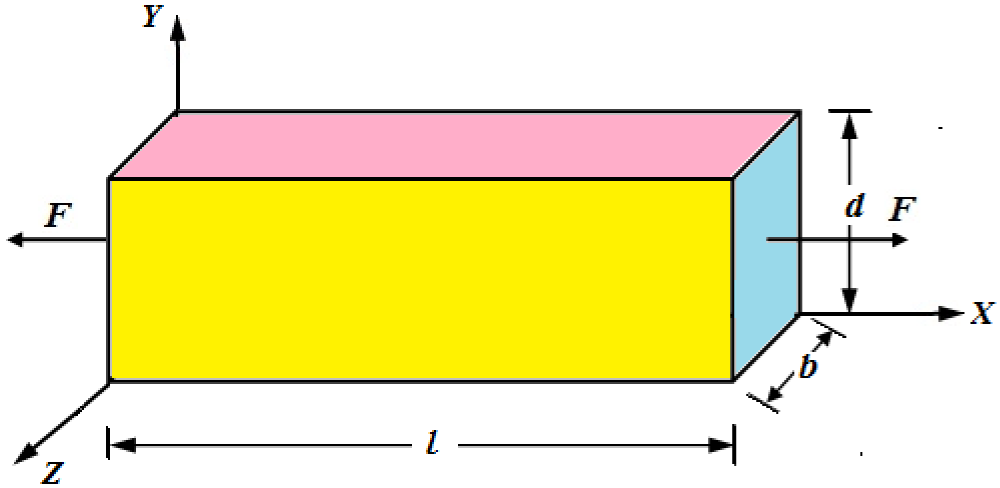


Fig. Axial load along X-axis

The value of for most materials varies from 0.2 to 0.33. Poisson's ratio for steel is 0.3 and that for concrete is 0.15. Poisson's ratio cannot be 0.5 for any material.

Let us consider a bar of length *l*, breath *b* and depth *d* subjected to axial tension of *F* along *X*-direction as shown in the figure.

Let strains along X, Y and Z directions be $\varepsilon_x, \varepsilon_y$ and ε_z respectively.

Mathematically,

$$\varepsilon_y = -\nu\varepsilon_x$$

$$\varepsilon_z = -\nu\varepsilon_x$$

Negative sign indicates that the strain in the Y and Z directions are compressive, i.e., opposite to the strain in X direction.

Volumetric strain:

It is the ratio of change in volume to original volume of the body (structural member) subjected to loading.

$$\varepsilon_v = \frac{\Delta V}{V}$$

Where,
ε_v = volumetric strain
V = original volume of the body
ΔV = Change in volume (increase or decrease)

Volumetric strain is the sum of strains in three mutually perpendicular directions.

Volumetric strain of a rectangular bar (cuboid):

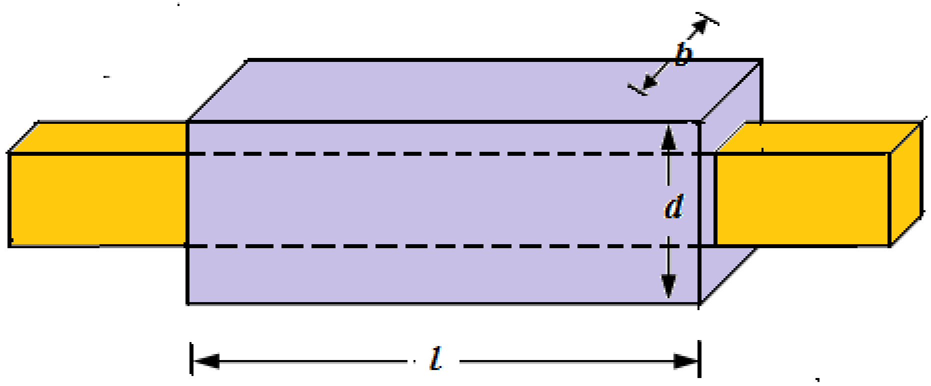


Fig.

Let us consider a rectangular bar of length *l*, breadth *b* and depth *d* is subjected to axial tension P_1, P_2 and P_3 along X, Y and Z directions respectively as shown in the figure.

Let, δl = change in length

δb = change in breadth

δd = change in depth

Original volume, $V = lbd$

Final volume = $(l + \delta l)\ (b + \delta b)\ (l + \delta d)$

$= lbd + lb\,\delta d + ld\,\delta b + bd\,\delta l + l\,\delta b\,\delta d + b\,\delta l\,\delta d + d\,\delta l\,\delta b + \delta l\,\delta b\,\delta d$

$= lbd + lb\,\delta d + ld\,\delta b + bd\,\delta l$ (neglecting product of small quantities)

Change in volume, δV = Final volume – Initial volume

$$= (lbd + lb\,\delta d + ld\,\delta b + bd\,\delta l) - lbd$$

$$= lb\,\delta d + ld\,\delta b + bd\,\delta l$$

$$\text{Volumetric strain} = \frac{\text{Change in volume}}{\text{Original volume}} = \frac{\delta V}{V} = \frac{lb\delta d + ld\delta b + bd\delta l}{lbd}$$

$$= \frac{\delta l}{l} + \frac{\delta b}{b} + \frac{\delta d}{d}$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Volumetric strain of a cylindrical rod:

Let us consider a cylindrical of length l and diameter d as shown in the figure subjected to axial pulls. Let the change in length and diameter is δl and δd respectively.

$$\textit{Original volume} = \frac{\pi d^2 l}{4}$$

$$\textit{Final volume} = \frac{\pi}{4}(d + \delta d)^2 (l + \delta l) = \frac{\pi}{4}\left(d^2 l + d\ \delta l + 2ld\delta d\right)$$

Ignoring products and higher powers of smaller quantities,

$$\text{Change in volume} = \delta V = \frac{\pi}{4}\left(d^2\delta l + 2ld\delta d\right)$$

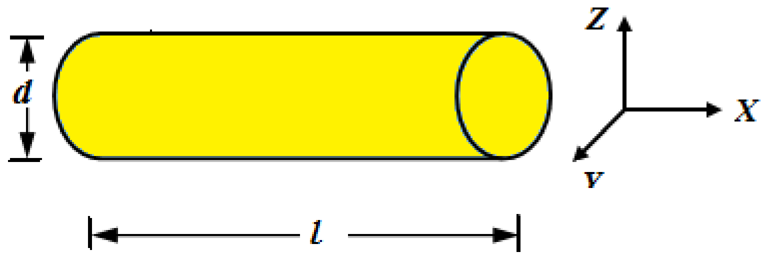
$$\text{Volumetric strain} = \varepsilon_v = \frac{\text{Change in volume}}{\text{Original volume}} = \frac{\delta V}{V}$$

$$= \frac{d^2\delta l + 2ld\delta d}{d^2 l} = \frac{\delta l}{l} + 2\frac{\delta d}{d}$$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Since, $$\varepsilon_y = \varepsilon_z = \frac{\delta d}{d}$$

Volumetric strain = Strain of the length + Twice the strain of the diameter.



In general, for any shape volumetric strain may be taken as the sum of the strains in three mutually perpendicular directions.

Volumetric strain of a sphere:

Let us consider a cylindrical of diameter *d* as shown in the figure. Let the change in length and diameter is *δd*.

$$Original\ volume, V = \frac{\pi d^3}{6}$$

$$Final\ volume = \frac{\pi}{6}(d + \delta d)^3$$

$$= \frac{\pi}{4}\left(d^3 + 3d^2\delta d\right) \quad igoring\ the higher\ powers\ of\ \delta d$$

$$\text{Change in volume} = \delta V = \frac{\pi}{6}\left(3d^2\delta d\right)$$

$$\text{Volumetric strain} = \varepsilon_v = \frac{\text{Change in volume}}{\text{Original volume}} = \frac{\delta V}{V}$$

$$= \frac{3d^2\delta d}{d^3} = \frac{3\delta d}{d} = 3\varepsilon_d$$

Volumetric strain = Three times the strain in the diameter.

Shear Stress: Stress induced/developed on a plane subjected to tangential force (force parallel to the plane) is called ***shear stress***. A material is said to be in a state of ***simple shear*** if it is subjected to only shearing stress.

If two equal and opposite parallel force act on a body, then there is a tendency of one part of the body to slide over the other. The stress induced in the body is known as shear stress. Shear stress is tangential to the area over which it acts.

A shearing stress alters only the shape of the body, leaving the volume unchanged.

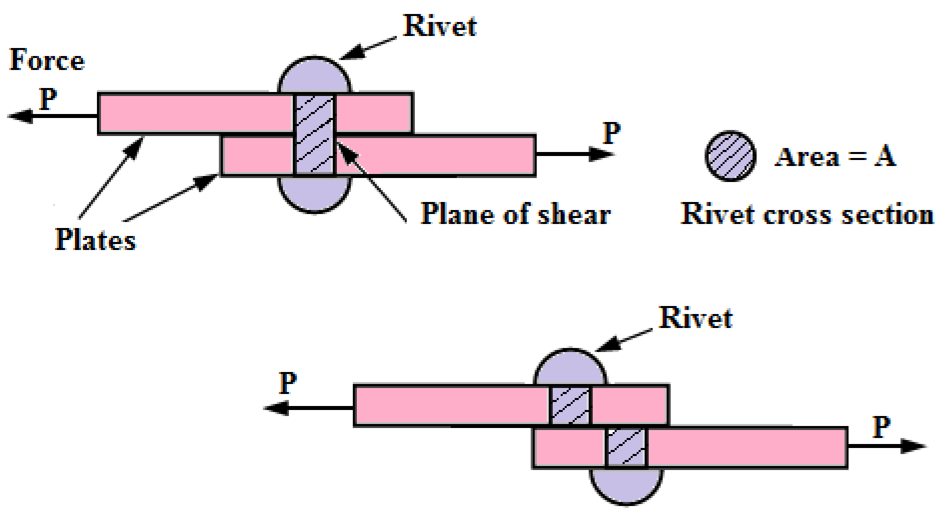


Fig. Shear stress

Let us consider a rectangular block of length *l*, width *b* and depth *d* is fixed to the surface at its bottom face as shown in the figure.

F = Force applied tangentially along surface AB and is called shear force

For equilibrium of the block the surface at CD will offer equal and opposite tangential reaction R = F.

Let the block be consists of two parts 1 and 2 separated by a section XX.

Consider the equilibrium of part 1. In order that the part 1 doesn't move from left to right, the part 2 will offer a resistance along the section XX such that R = F. Similarly, for equilibrium of part 2, part 1 will offer a resistance R along the section XX such that R = F.

The resistance R along the section XX is called shear resistance.

$$Shear\ stress, \tau = \frac{\text{Shear resistance}}{\text{Shear area}} = \frac{R}{A} = \frac{F}{A}$$

$$\tau = \frac{F}{l \times b}$$

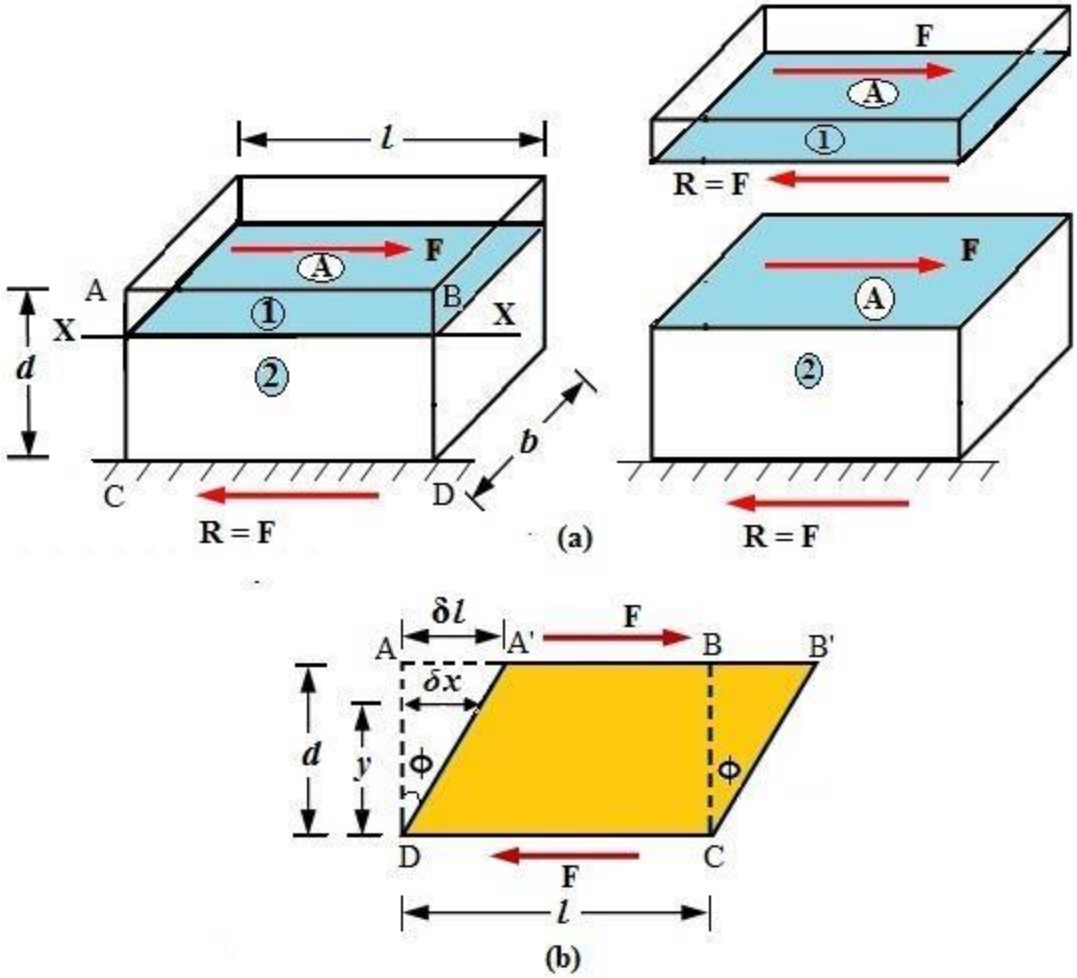


Fig. Shear (a) stress and (b) strain

Shear deformation:

When the block doesn't fail in shear, it undergoes shear deformation as shown in figure (b). Since the bottom of the block is fixed, the block takes the shape of A'B'CD. The block undergoes distortion of ADA' = ϕ.

Let us think the block comprises of a number of thin horizontal layers. Each layer undergoes horizontal displacement in proportion to its distance from the lower face of the block. The ratio of the horizontal displacement of the layer to distance of the layer from the bottom face is known as shear strain.

$$Shear\ strain = \frac{\delta l}{d}$$

$$\tan\phi = \frac{\delta l}{d}$$

$$\text{When } \phi \text{ is very small, } \phi \cong \tan\phi = \frac{\delta l}{d}$$

Elastic Limit: The limit of stress within which a structural member of any material under stress regains its original shape and size after removal of loading is called ***elastic limit*** of the material.

The maximum stress which the material can withstand without causing permanent deformation after the removal of load is called elastic limit.

If the stress exceeds the elastic limit, the member will not regain its original configuration. A residual strain called ***permanent set*** remains in the member.

Limit of proportionality: The limiting value of stress up to which stress is proportional to the strain is known as ***limit of proportionality***.

In case of steel, a well pronounced proportionality limit can be found from tensile test. Some materials have a very small value of proportionality limit, others may not show such limit.

Hooke's Law: In 1678, English Mathematician Robert Hooke observed that there is a definite relationship between the elastic deformation (measured as *strain*) and the stress intensity causing it. He offered his observation in the form of a law called Hooke's law.

Hooke's law states that when a material is loaded up to certain limit of stress intensity within elastic limit, the stress is directly proportional the strain in the material.

Mathematically,

$$\sigma \propto \varepsilon$$

$$\Rightarrow \frac{\sigma}{\varepsilon} = E \text{ (Constant of proportion ality)}$$

Where, σ is stress and ε is strain in the material of the member. The constant of proportionality, E is known as ***modulus of elasticity*** *or* ***elastic* modulus**. The elastic modulus for normal stress and strain is also called Young's modulus. The modulus of elasticity is a measure of stiffness of the material. It has the same unit as stress. The slope of the stress-strain diagram in the linearly elastic region gives the value of ***modulus of elasticity*** (*E*). Modulus of elasticity of steel is equal to 210 GPa, aluminum is 73 GPa and plastic is from 1 to 1.4 GPa. (1 GPa = 1000 N/mm^2)

If *P* is the axial force acting in a prismatic member of sectional area *A* and length *l,*

$$Strain, \quad \frac{\delta l}{l} = \varepsilon \Rightarrow \delta l = \varepsilon l$$

$$\Rightarrow \delta l = \frac{\sigma l}{E} = \frac{Pl}{AE}$$

The product *AE* is called the ***axial rigidity*** or ***axial stiffness*** of the member.

For axial member, force necessary to produce one unit deformation (deflection) is known as ***axial stiffness*** and is denoted by *K*. Hence, AE/L is the axial stiffness for axially loaded member of length *L*. Reciprocal of stiffness, $1/K$ is known as flexibility. More is the value of K, more is the stiffness and less is the flexibility and vice versa.

Modulus of rigidity: The ratio of shearing stress to corresponding shear strain within elastic limit is a constant known as modulus of rigidity or shear modulus. It is denoted by letter *G*. It has same unit as shear stress.

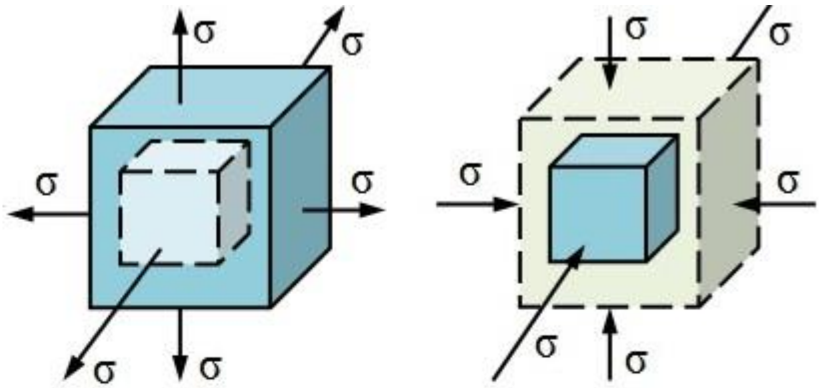
$$\text{Modulus of rigidity,} \quad G = \frac{\tau}{\phi}$$

Where,

τ = *Shear stress*

and ϕ = *Shear strain*

Bulk modulus: When a body is subjected to identical, σ (equal and like) stresses in three mutually perpendicular directions, it undergoes uniform changes in three directions without undergoing distortion of shape. In such case, the ratio of normal stress to volumetric strain is called ***Bulk modulus***. It is denoted by letter *K*.



(a) Identical tensile stress (b) Identical compressive stress

Fig. Stresses in three mutually perpendicular directions

***Bulk modulus** may be defined as the ratio of identical pressure acting in three mutually perpendicular directions to corresponding volumetric strain.*

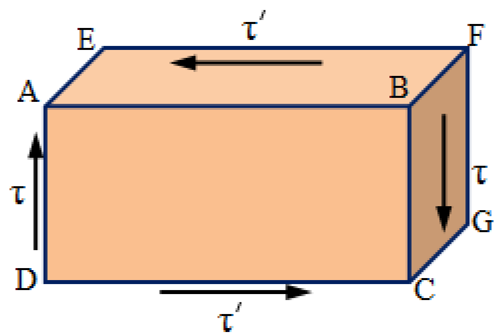
The element in figure (a) is subjected to identical tensile stress in three mutually perpendicular directions and the element in figure (b) is subjected to identical compressive stresses. Hydrostatic pressure acting on a submerged body is an ideal example of identical stresses in three mutually perpendicular directions.

Mathematically, $$K = \frac{\sigma}{\varepsilon_v}$$

Complimentary shear stress:

A set of shear stresses acting across a plane is always accompanied by a balancing set of transverse shear stresses of same intensity across a plane normal to the previous plane. The balancing stress is called ***complimentary shear stress***.

Let us consider a rectangular block ABCD of unit thickness perpendicular to the plane of the paper subjected to a shearing stress, τ alongside AB and CD as displayed in the figure. The forces acting on these two faces are each equal to τ.(AB.1) = τ.(CD.1). These two equal, opposite and parallel forces will form an anti-clockwise couple of magnitude = τ x AB x AD.



If the block is in equilibrium, a restoring (clockwise) couple of equal magnitude has to be developed by virtue of material resistance of the block. For this to happen, shear stress of intensity τ' must be set upon the faces AD and BC.

The forces acting on these faces are each equal to τ'.AD. These two equal, opposite and parallel forces will form a restoring couple (clockwise) of magnitude = τ' x AD x AB.

For, equilibrium equating the moment of two couples (acting and restoring), we have

$$\tau'.AD.AB = \tau.AB.AD$$
$$\Rightarrow \tau' = \tau$$

τ' is called the compliment aryshear stress.

Hence, a set of shear stresses is always accompanied by a complimentary set of shear stresses of equal intensity.

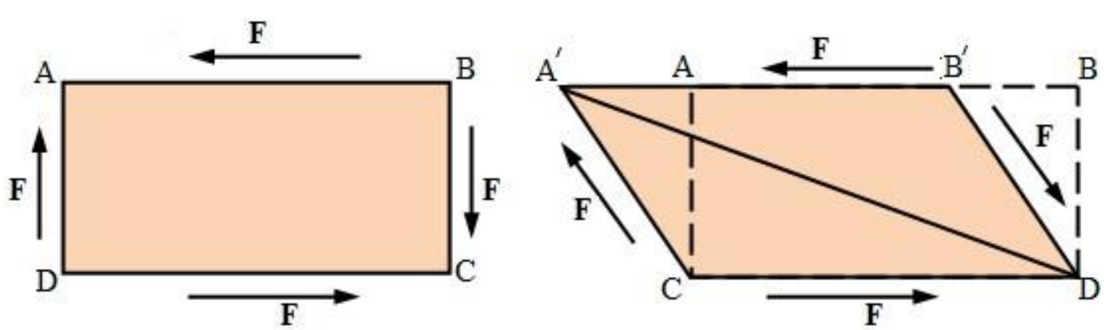


Fig. Complimentary shear stress

As a result of the two couples formed due to applied shear forces and the shear forces arising due to complimentary shear stresses, the diagonal AC of the block will be subjected to tension while the diagonal BD will be subjected to compression.

Diagonal tension and compression:

Consider an elemental rectangular block ABCD with unit thickness perpendicular to the plane of the figure. Let the element be in the state of simple shear with shear stress intensity of τ across the surfaces as shown in the figure.

Consider the plane through the diagonal BC which makes an angle of θ with face CD. Consider the equilibrium of triangular wedge BCD. The wedge is subjected to the following forces.

a) Force along BC = τ.BC (↓)
b) Force along DC = τ.DC (→)
c) Force normal to the plane DB = σ_θ . BD
d) Force tangential to the plane DB = τ_θ .BD

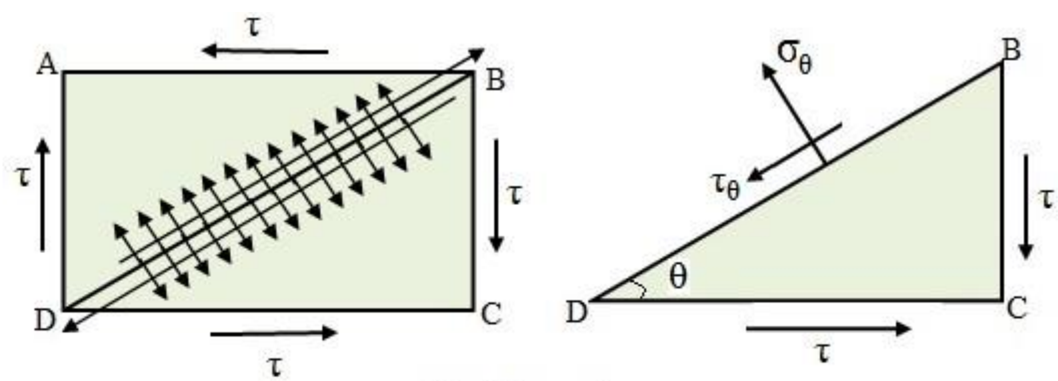


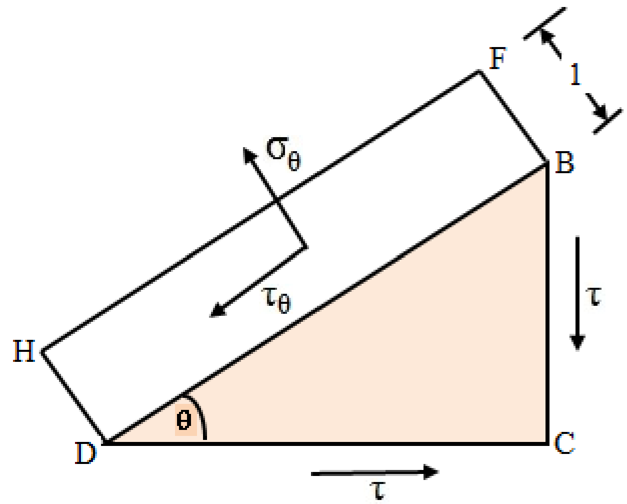
Fig. Diagonal

Resolving the forces normal to the plane BD and along the plane BD, we have

$$\sigma_\theta \left(BD.1\right) = \tau \left(BC.1\right)\cos\theta + \tau \left(DC.1\right)\sin\theta$$

$$\Rightarrow \sigma_\theta .BD = \tau .BD \sin\theta \cos\theta + \tau .BD \cos\theta \sin\theta .$$

$$\Rightarrow \quad \sigma_\theta = \tau .2\sin\theta \cos\theta = \tau \sin 2\theta$$



$$\tau_\theta (BD.1) = -\tau (BC.1)\sin\theta + \tau (DC.1)\cos\theta$$

Again, $$\Rightarrow \tau_\theta .BD = -\tau .BD \sin\theta \sin\theta + \tau .B \cos\theta \cos\theta$$

$$\Rightarrow \quad \tau_\theta = \tau .\left(\cos^2\theta - \sin^2\theta\right) = \tau \cos 2\theta$$

Hence, normal and tangential stresses on plane BD are

$$\sigma_\theta = \tau \sin 2\theta$$
$$and \quad \tau_\theta = \tau \cos 2\theta$$

For plane of maximum normal stress,

$$\sin 2\theta = \pm 1$$
$$i.e., 2\theta = \pm 90^0 \quad i.e., \theta = \pm 45^0$$
$$when \theta = 45^0, \sigma_\theta = +\tau \quad (+ve sign\ indicates\ that\ normal\ stress\ is\ tensile)$$
$$when \theta = -45^0, \sigma_\theta = -\tau \quad (-ve sign\ indicates\ normal\ stress\ compressive)$$

Corresponding to $\theta = \pm 45^0$, tangential stress $\tau_\theta = 0$

Hence, the planes carrying maximum normal stress don't carry any shear stress.

For plane of maximum shear stress,

$$\cos 2\theta = \pm 1$$
$$i.e., 2\theta = 0\ or\ 180^0 \quad i.e., \theta = 0\ or\ 90^0$$

Planes corresponding to maximum shear stress, the normal stresses are zero.

When an elemnt is in the state of simple shear, the maximum direct stresses are induced on mutually perpendicular planes which are at 45^0 to the planes of pure shear. One of the maximum direct stresses is tensile while the other is compressive. The direct maximum tensile and maximum compressive stresses are of the same magnitude as the shear stress on the planes of pure hear.

Relationship between modulus of elasticity and modulus of rigidity:

Consider a square block ABCD of side *a* and thickess unity perpendicular to the plane of the paper. Let the block be in the state of simple shear (pure shear) with shear stress intensity of τ across the surfaces as shown in the figure. The block will undergo distortion of shape due the system of stresses.

It is evident from the stress components that the diagonal AC will be elongated and the diagonal BD will be shortened.

The increase in length of the diagonal AC can be computed by considering the diagonal tensile and compressive stresses on AC and BD respectively. It is known that the diagonal tensile and compressive stresses in a square block are each equal to τ.

Strain in the diagonal AC = Strain in AC due to diagonal tensile stress on the plane BD + Strain in length AC due to diagonal compressive stress on plane AC.

$$Strain\,in\,AC = \frac{\tau}{E} + \frac{\nu.\tau}{E} = \frac{\tau}{E}(1+\nu) \qquad (1)$$

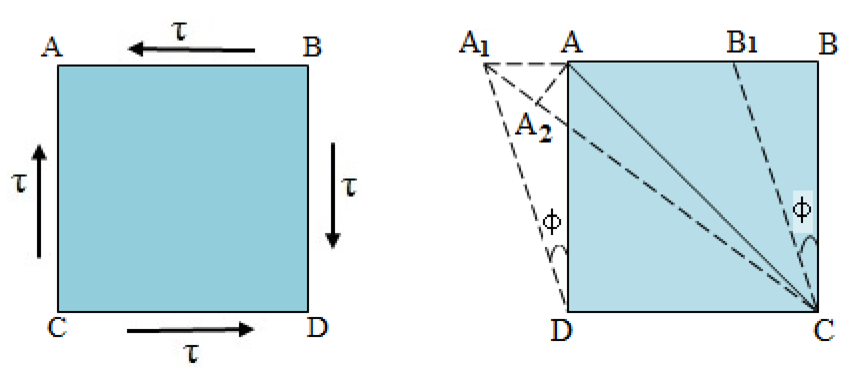


Fig. Square block with simple shear

Strain in the diagonal AC can also be computeed from the distorted geometry of the block.

Let the geometry ABCD deform to the position A_1B_1CD through the angle ϕ.

Increase in length of the digonal AC = A_1C – AC.

Let AA_2 be perpendicular to A_1C. Since the amgle ACA_2 is very small, AC = A_2C

Hence, icrease in length of the diagonal AC = A_1C - A_2C = A_1A_2 = $AA_1\cos AA_1A_2$. But the angle AA_1A_2 is equal to BAC = 45^0.

Hence, increase in length of the diagonal AC

$$= AA_1 \cos 45^0 = \frac{AA_1}{\sqrt{2}}$$

$$\text{Shear strain, } \phi = \frac{AA_1}{AD} = \frac{AA_1}{a}$$

$$\Rightarrow AA_1 = a\phi$$

Icrease in length of the diagonal $AC = a\sqrt{2}$.

Length of the diagonal $AC = a\sqrt{2}$

$$\text{Strain of the diagonal } AC = \frac{\text{Increase in length}}{\text{Original length}} = \frac{a\phi}{\sqrt{2}} \cdot \frac{1}{a\sqrt{2}} = \frac{\phi}{2} \qquad (2)$$

From Equation (1) and (2), we have

$$\frac{\phi}{2} = \frac{\tau}{E}(1+\nu) \Rightarrow \frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$$

$$\therefore \quad E = 2G(1+\nu) \qquad (3)$$

Relationship between Young's modulus and bulk modulus:

Consider a cube of side *'a'* subjected to direct tensile stress of intensity σ in three mutually perpendicular directions. Let E be the Young's modulus and ν, the Poisson's ratio.
Let us consider the strain in one edge AB of the cube.

Strain in AB due to tensile stress in the X-direction

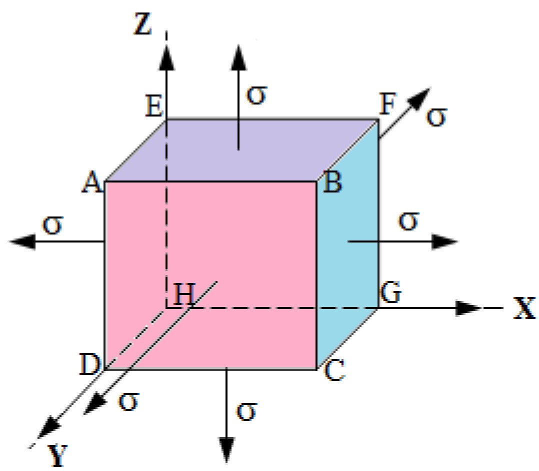


Fig.

$$\text{Strain in AB due to tensile stress in X-direction} = \frac{\sigma}{E}$$

$$\text{Strain in AB due to tensile stress in Y-direction} = -\frac{\nu\sigma}{E}$$

$$\text{Strain in AB due to tensile stress in Z-direction} = -\frac{\nu\sigma}{E}$$

$$\text{Total Strain in AB}, \varepsilon_x = \frac{\sigma}{E} - \frac{\nu\sigma}{E} - \frac{\nu\sigma}{E} = \frac{\sigma}{E}(1-2\nu)$$

Similarly strains in the other two edges are each qual to $\varepsilon_y = \varepsilon_z = \frac{\sigma}{E}(1-2\nu)$

$$\textit{Volumetric strain}, \varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{3\sigma}{E}(1-2\nu)$$

$$\text{Bulk modulus}, K = \frac{\text{Normal stress}}{\text{Volumetric train}} = \frac{\sigma}{\frac{3\sigma}{E}(1-2\nu)}$$

$$\Rightarrow E = 3K(1-2\nu) \qquad (4)$$

The relationship between Young's modulus and Bulk modulus is given by the above equation.

Further from equation (3), we have

$$E = 2G(1+\nu)$$
$$\Rightarrow 2\nu = \frac{E}{G} - 2$$

From equation (4), we have

$$E = 3K(1-2\nu)$$
$$\Rightarrow E = 3K\left\{1-\left(\frac{E}{G}-2\right)\right\}$$
$$\Rightarrow E = 3K\left(3-\frac{E}{G}\right) = \frac{9KG-3EK}{G}$$
$$\Rightarrow EG = 9KG - 3EK$$
$$\Rightarrow E(3K+G) = 9KG$$
$$\Rightarrow E = \frac{9KG}{3K+G}$$

The above equation is the relationship between three elastic modulii.

For isotropic and homogeneous material, there are four elastic constants, namely *E, G, K* and *v*. Out of the four constants two are independent constants and other two can be obtained by using the relationship between them.

Deformation of prismatic bar due to uniaxial loading:

Consider a prismatic bar subjected to uniaxial tensile load as shown in the figure.

Let

P = Load acting on the bar
l = Original length of the bar
A = Cross - sectional area of the bar
σ = Stress induced in the bar
E = Young's modulus of the material of the bar
ε = Strain in the ba
δl = Deformatio n or change in length of the bar

We know that,

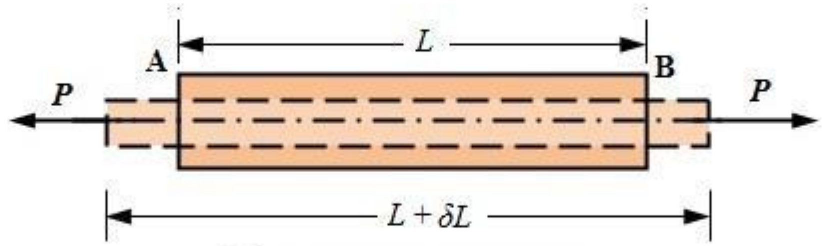


Fig. Uniaxial load (Tensile)

$$Stress, \sigma = \frac{P}{A}; \qquad Strain, \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

$$and\ deformation, \delta l = \varepsilon . E = \frac{\sigma . l}{E} = \frac{Pl}{AE}$$

Deformation of prismatic bar under self weight:

Consider a freely hanging prismatic bar *AB* of length *L*, cross sectional area *A* and weight *W* as displayed in the figure. Let *ω* = Specific weight of the material of the bar.

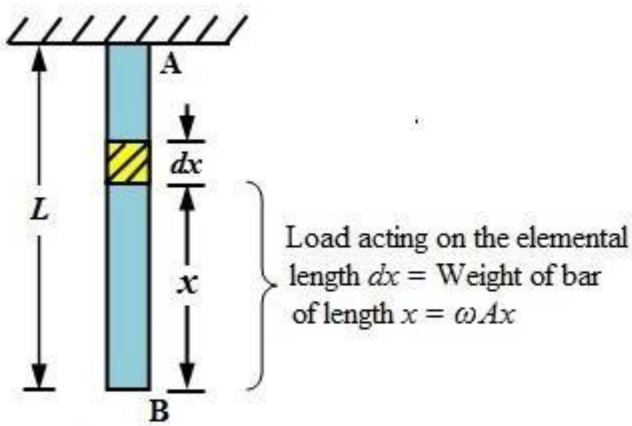


Fig.

Let us consider an elementary length, *dx* of the bar at a height of *x* from the bottom *B* of the bar. Weight of the portion of the bar below *XX* act as tensile force on the elemental length. Weight of the portion below *XX*, *P* = *ωAx*.

Elongation of the elementary length *dx* due to the weight of the bar of length *x*

$$\delta(dx) = \frac{P.dx}{AE} = \frac{(\omega Ax)dx}{AE}$$

$$= \frac{\omega x dx}{E}$$

Total elongation of the bar due to self weight,

$$\int_0^L \delta(dx) = \int_0^L \frac{\omega x dx}{E} = \frac{\omega}{E}\int_0^L x dx$$

$$= \frac{\omega}{E}\left[\frac{x^2}{2}\right]_0^L = \frac{\omega L^2}{2E}$$

$$= \frac{(\omega LA).L}{2AE}$$

$$\Rightarrow \delta L = \frac{WL}{2AE}$$

Tensile Test:

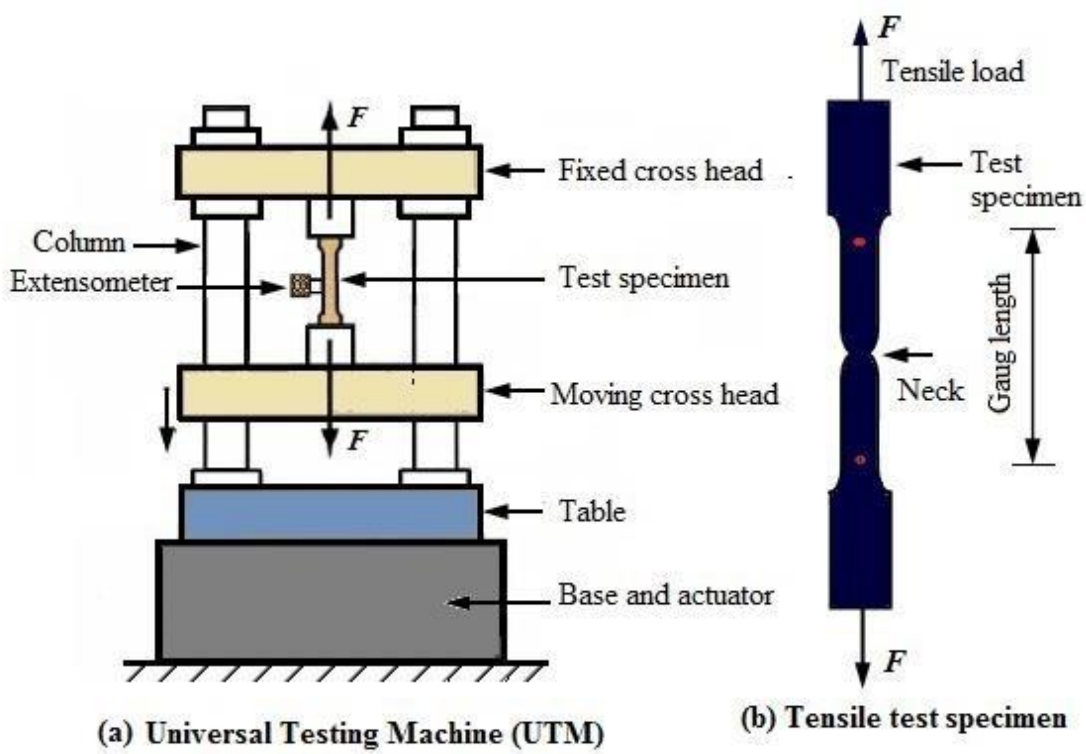


Fig. Tensile Test Setup

The **tension test** is the most common method for determining the *mechanical properties* of materials, such as strength, ductility, toughness, elastic modulus, and strain hardening capability. In a tension test, a specimen of standard dimension is subjected to a continually increasing uniaxial tensile force while simultaneous observation of elongation is taken by means of extensometer fitted to the specimen. The results of the test are plotted as ***stress-strain diagram*.** A standard tensile test set up with the representative (a) UTM (Universal Testing Machine) and (b) tensile test specimen is shown in the figure.

A ***stress-strain diagram*** is a diagram in which corresponding values of stress and strain are plotted against each other. The values of the stress are plotted as ordinates in vertical (Y-axis) and values of strain as abscissas in horizontal axis (X-axis).

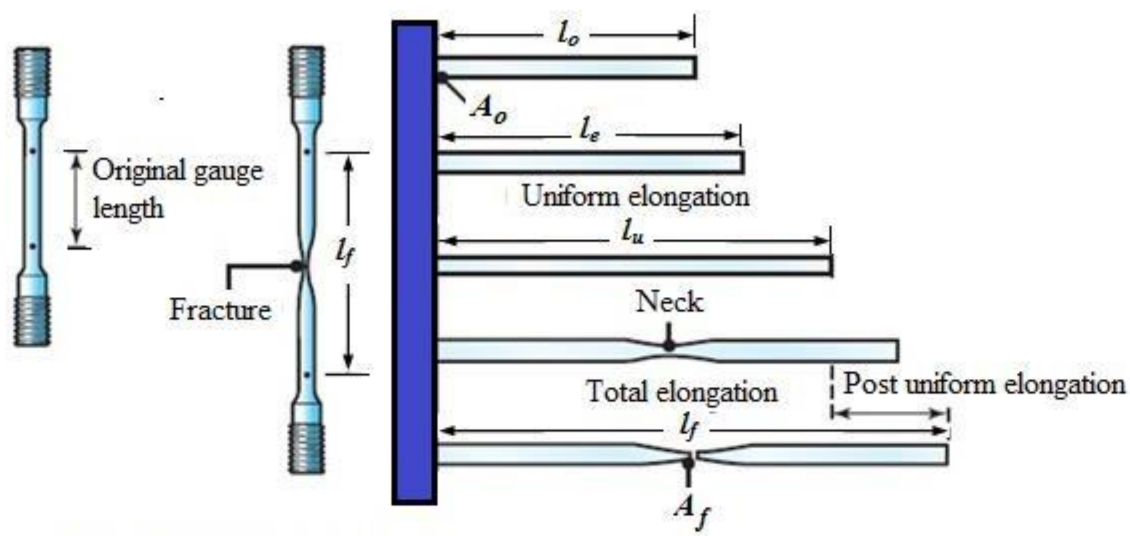


Fig. Typical diagram of tensile testing material specimen

Stress-strain curve for ductile material:

A representative/typical stress-strain (σ-ε) diagram for ductile material (mild steel) is displayed in the figure with salient points on the curve.

Stress is proportional to the strain up to the point *A*, i.e., the stress-strain variation is linear. This point represents ***limit of proportionality***. Hooke's law holds good up to this point. Slope of stress-strain line from *O* to *A* gives the modulus of elasticity also known as Young's modulus.

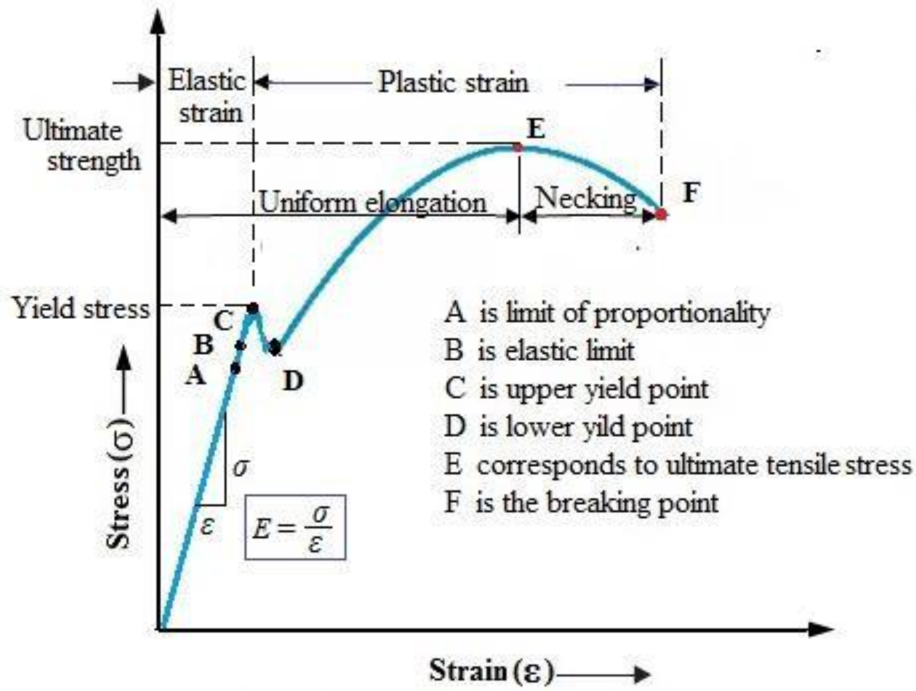


Fig. Stress-strain diagram for ductile material

Beyond point *A* and up to point *B*, material remains elastic *i.e.,* the material returns to its original condition of the forces acting on it is removed. The stress corresponding to represents the ***stress at elastic limit***. If the specimen is stressed beyond point *B*, permanent set takes place and we enter plastic deformation region. In the plastic deformation region, the strain does not get fully removed even with the removal of the force causing it.

If the force is increased further, point '*C*' is reached where the test specimen elongates even when the stress is not increased. This point is called yield point. In fact, there are two yield points *C* and *D* which are called upper and lower yield points respectively. The stress corresponding to the yield point is known as ***yield stress or yield strength***.

With further straining, the effect of a phenomenon called strain hardening or work hardening takes place.* The material becomes stronger and harder and its load bearing capacity increases. The test specimen is therefore able to bear more stress. On progressively increasing the force acting on the specimen, point *E* is reached. This point is the highest point in the stress-strain curve and represents the point of maximum stress. It is, therefore, called ***ultimate tensile strength*** (UTS) of the material. It is equal to the maximum load applied divided by the original cross-sectional area (A_0) of the test specimen.

After UTS point *E*, a sharp reduction in cross-sectional area of the test specimen takes place and a **"neck"** is formed in the centre of the specimen. Ultimately the test specimen breaks in two pieces as the neck becomes thinner and thinner. The point *F* represents the breaking point and the corresponding stress is known as ***breaking stress or fractured stress***. The actual breaking stress is much higher than the UTS, if the reduced cross-sectional area of the test specimen is taken into account.

As plastic deformation increases, the cross-sectional area of the specimen decreases. However for calculation of the stress in the stress-strain graph, the original cross-sectional area is considered. It is for this reason that the breaking point, *F* seems to occur at a lower stress level than the UTS point *E*.

The measure of the strength of a material is the ultimate tensile strength (σ at point *E*). However, from the design point of view, the yield point is more important as the designed structure should withstand forces without yielding. Usually yield stress (σ at point *D*) is two-thirds of the UTS and this is referred to as ***yield-strength*** of the material.

Stress-strain curve for brittle material:

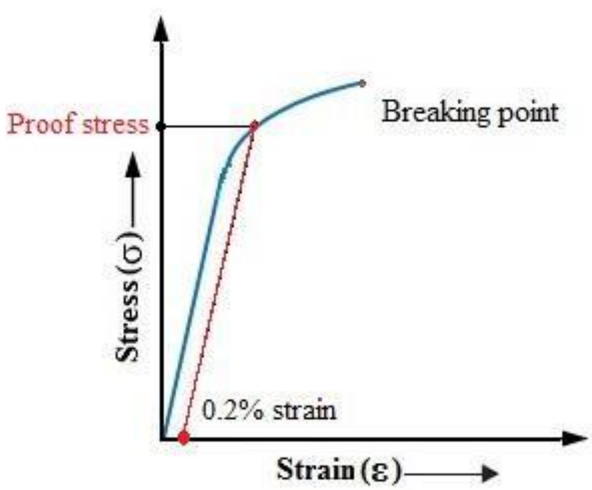


Fig. Stress-strain curve for brittle material

A stress-strain curve for brittle material (cast iron) is obtained by subjecting a test bar of such material in a tensile testing machine. The tensile load is gradually increased and the extension of the test piece is recorded. The stress-strain curve for a brittle material is significantly different from that for a ductile material. A typical stress-strain curve for a brittle material is shown in Fig.

This curve displays no yield point, and the test specimen breaks suddenly without any appreciable necking or extension. In the absence of a yield point, concept of ***"proof-stress"*** has been evolved for measuring yield strength of a brittle material. For example, 0.2% proof-stress indicates the stress at which the test specimen 'suffers' a permanent elongation equal to 0.2% of initial gauge length.

Percentage of elongation: Increase in length of tensile test sample expressed in percentage of original length of the specimen is known as percentage of elongation or percentage of increase in length. It has considerable significance in engineering because it indicates the ductility of the material.

The ability of material to deform appreciably without rupture is known as ductility. It is a measure of the amount of plastic deformation that a material goes through before it fails. It refers to plastic deformation under tensile loads. Ductility enables the material to be drawn into wires. Highly ductile metals can exhibit significant strain before fracturing, whereas brittle materials frequently display very little strain. Ductility increases with temperature.

There are two common measures of ductility.

The first is the **total elongation** of the specimen, given by

$$\text{Percentage elongation } (\%) = \frac{(l_f - l_0)}{l_0} \text{x}100$$

where l_f and l_0 are the original and final length of the test specimen.

The second measure of ductility is the reduction of area, given by

$$\text{Reduction of area } (\%) = \frac{(A_f - A_0)}{A_0} \text{x}100$$

where, A_f and A_0 are, respectively, the original and final (fracture) cross-sectional area of the test specimen.

Reduction of area and elongation are generally interrelated. Thus, the ductility of a piece of chalk is zero, because it does not stretch at all or reduce in cross section; by contrast, a ductile specimen, such as putty or chewing gum, stretches and necks considerably before it fails.

Metals with more than 15% elongation at fracture are considered as ***ductile***. Metals with 5 to 15% elongation are considered of ***intermediate ductility***. However, the metals with less than 5% elongation, i.e., strain of 0.05 are considered as ***brittle*** ones. Brittle materials include ceramic, glass and some alloys. Cast iron is also classified as brittle.

Working stress: The maximum stress to which a structural member is ever allowed to subjected to is called ***working stress***. It should be below the elastic limit.

Ultimate stress: The maximum stress to which the material of the test piece is subjected to during the test is known as ***ultimate stress*** or ***ultimate strength***. It is obtained by the maximum load to which the test piece is subjected to divided by the original cross-sectional area.

Factor of safety: the ratio of the ultimate stress to the working stress is called the ***factor of safety***. The value of factor of safety in engineering design varies from 3 (for accurately known dead load) to 12 (for shock loads of indefinite magnitude).

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Working stress}}$$

In case of ductile material, since excessive deformation creates problem in the performance of the structural member, working stress is taken as a factor of yield stress or that of proof stress (if yield stress doesn’t exists) in place of ultimate stress.

Factor of safety for steel is 1.85, for concrete it is 3.

True stress: it is the ratio of the load to the actual cross-sectional area of the test piece.

Engineering stress: It is the ratio of the load to the original cross-sectional area of the test piece. It is also known as ***nominal stress***.

Numerical Examples

1. An elastic rod 25 *mm* diameter, 200 *mm* long extends by 0.25 *mm* under a tensile load of 40 *kN*. Find the intensity of stress, strain and elastic modulus for the material of the rod.

Solution. Given, Diameter of rod, $d = 25\ mm$, Length, $l = 200\ mm$

Increase in length, $\delta l = 0.25\ mm$, Load, $P = 40\ kN = 40{,}000\ N$

$$\text{Area, A} = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 25^2 = 490.87\ mm^2$$

$$\text{Intensity of stress, } \sigma = \frac{\text{Load}}{\text{Area}} = \frac{P}{A} = \frac{40000\ N}{490.87\ mm^2} = 81.49\ N/mm^2$$

$$\text{Strain, } \varepsilon = \frac{\delta l}{l} = \frac{0.25}{200} = 0.00125$$

$$\text{Elastic modulus, } E = \frac{\sigma}{\varepsilon} = \frac{81.49}{0.00125} = 65{,}192\ N/mm^2$$

2. A steel rod of 25 *mm* diameter and 2 *m* long is subjected to an axial pull of 45 *kN*. Find the (a) intensity of stress, (b) strain and (c) elongation. Take $E = 2 \times 10^5\ N/mm^2$.

Solution. Given, Diameter of rod, $d = 25$ mm, Length, $l = 2$ m = 2000 mm

Load, $P = 45\ kN = 45{,}000\ N$

$E = 2 \times 10^5\ N/mm^2$

$$\text{Area, } A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 25^2 = 490.87\ mm^2$$

$$\text{Intensity of stress, } \sigma = \frac{\text{Load}}{\text{Area}} = \frac{P}{A} = \frac{45{,}000\ N}{490.87\ mm^2} = 91.67\ N/mm^2$$

$$\text{Strain, } \varepsilon = \frac{\sigma}{E} = \frac{91.67}{2 \times 10^5} = 4.583 \times 10^{-4} = 0.0004583$$

$$\text{Elongation} = \text{Strain} \times \text{Original length} = \varepsilon \times l = 4.583 \times 10^{-4} \times 2000$$
$$= 0.916\ mm$$

3. A load of 4000 *N* has to be raised at the end of a steel wire. If the unit stress in the wire must not exceed 80 N/mm^2, what is the minimum diameter of the rod required? What will be the extension of 3.5 *m* length of the wire? Take $E = 2 \times 10^5\ N/mm^2$.

Solution. Given, Length, $l = 3.5$ m = 3500 mm, Permissible stress = 80 N/mm^2

Load to be raised, $P = 4000\ N$, $E = 2 \times 10^5\ N/mm^2$

Let the minimum required diameter of rod = d

$$\text{Cross-sectional Area, } A = \frac{\pi}{4} d^2$$

$$\text{Stress in the wire, } \sigma = \frac{P}{A} = \frac{4{,}000}{\left(\frac{\pi}{4}\right) d^2}$$

For the wire to sustain the load, $\sigma \leq$ Permissible stress

$$\Rightarrow \quad \frac{4{,}000}{\left(\frac{\pi}{4}\right) d^2} \leq 80$$

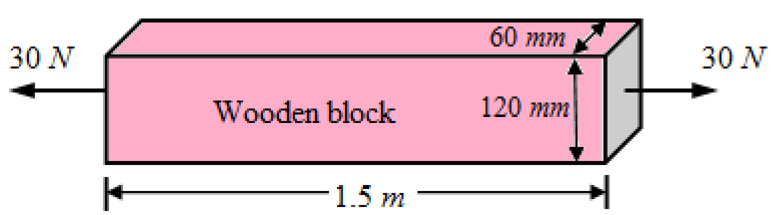
$$\Rightarrow \quad \frac{\pi}{4} d^2 \geq \frac{4000}{40}$$

$$\Rightarrow \quad d \geq \sqrt{\frac{400}{\pi}}$$

$$\Rightarrow \quad d \geq 7.97\ mm$$

$$\text{Elongation}, \delta l = \frac{\sigma . l}{E} = \frac{80 \times 3500}{2 \times 10^5} = 1.44\ mm$$

4. A wooden tie in the figure is 60 *mm* wide, 120 *mm* deep and 1.5 *m* long. It is subjected to an axial pull of 30 *N*. The stretch of the member is found to be 0.625 *mm*. Find the Young's modulus of the material.



Solution. Given, Length, $l = 1.5\ m = 1500\ mm$, Width = 60 *mm*

Depth = 120 *mm*, Pull, $P = 30\ N$

Increase in length, $\delta l = 0.625\ mm$

$$\text{Area of cross-section} = Width \times Depth = 60 \times 120$$
$$= 7200\, mm^2$$
$$\text{Young's modulus,} \quad E = \frac{PL}{A.\delta l} = \frac{30 \times 1500}{7200 \times 0.625} = 10\, N/mm^2$$

5. A hollow steel column of external diameter 250 *mm* has to support an axial load of 2000 *kN*. If the ultimate stress for the steel column is 480 N/mm^2, find the internal diameter of the column allowing a load factor of 4.

Solution. Given, External diameter, D = 250 *mm*, Axial load, P = 2000 kN

Ultimate stress, $\sigma_u = 480\ N/mm^2$, Load factor = 4

Let the internal diameter of rod = *d*

$$\text{Load factor} = \frac{\text{Ultimate stress}}{\text{Safe stress}}$$

$$\Rightarrow Safe stress = \frac{480}{4} = 120\, N/mm^2$$

$$Safe stress = \frac{P}{A} = \frac{2000 \times 1000}{\frac{\pi}{4} \times \left(250^2 - d^2\right)}$$

$$\Rightarrow \quad 120 = \frac{8 \times 10^6}{\pi\left(250^2 - d^2\right)}$$

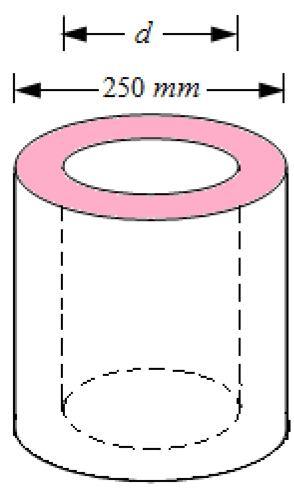
$$\Rightarrow \quad \left(250^2 - d^2\right) = \frac{8 \times 10^6}{120\pi}$$

$$\Rightarrow \quad 250^2 - d^2 = 21220.659$$

$$\Rightarrow \quad d^2 = 250^2 - 21220.659$$

$$\Rightarrow \quad d^2 = 41279.341$$

$$\Rightarrow \quad d = 203.173\, mm$$



6. The following data refers to the tensile test conducted on a mild steel bar.
   i. Diameter of steel bar = 3 *mm*
   ii. Gauge length = 200 *mm*
   iii. Extension at a load of 100 *kN* = 0.139 *mm*
   iv. Load at elastic limit = 230 *kN*

v. Maximum load = 360 kN

vi. Total extension = 56 mm

vii. Diameter of rod at failure = 22.25 mm

Calculate (a) Young's modulus, (b) the stress at elastic limit, (c) the percentage of elongation and (d) the percentage decrease in area

Solution. Gauge length, l_o = 200 mm, Diameter of the bar, d_o = 30 mm

a) Young's modulus.

$$\text{Cross-sectional area, } A = \frac{\pi d_o^2}{4} = \frac{\pi \times 30^2}{4} = 706.86\,mm^2$$

$$\text{Stress at load } (P = 100\,kN), \sigma = \frac{P}{A} = \frac{100 \times 1000}{706.86} = 141.47\,N/mm^2$$

$$\text{Strain at load } (P = 100\,kN), \varepsilon = \frac{\text{Extension}}{\text{Original length}} = \frac{0.139}{200} = 0.000695$$

$$\text{Young's modulus, } E = \frac{\sigma}{\varepsilon} = \frac{141.47}{0.000695} = 2.035 \times 10^5\ N/mm^2$$

b) Stress at elastic limit.

Load at elastic limit, P_e = 230 kN

$$\text{Stress at elastic limit, } \sigma_e = \frac{P_e}{A} = \frac{230 \times 1000}{706.86} = 325.383\,N/mm^2$$

c) Percentage of elongation.

$$\text{Percentage of elongation} = \frac{\text{Total increase in length}}{\text{Gauge length}} \times 100 = \frac{56}{200} \times 100 = 28\%$$

d) Percentage decrease in area.

Diameter of rod at failure, d_f = 22.5 mm

$$\text{Decrease in area} = \frac{\pi}{4}(d_0^2 - d_f^2) = \frac{\pi}{4}\left(30^2 - 22.5^2\right)$$

$$= 309.250\,mm^2$$

$$\text{Percentage decreae in area} = \frac{\text{Decrease in area}}{\text{Gauge area}} \times 100 = \frac{309.25}{706.86} \times 100$$

$$= 43.74\%$$

7. A steel rod of 28 *mm* diameter and 300 *mm* long is subjected to axial forces alternating between a maximum compression of 16 *kN* and a maximum tension of 7 *kN*. Find the difference between the greatest and least lengths of the rod. Take $E = 210$ *GPa*.

Solution. Given: Diameter of rod, $d = 28$ *mm*; Length of the rod, $l = 300$ *mm*

Axial compression, $P_c = 16\ kN = 16000\ N$

Axial tension, $P_t = 7\ kN = 7000\ N$

$E = 210\ GPa = 210 \times 10^3\ N/mm^2$

$$\text{Area of cross-section, } A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 28^2 = 615.75\ mm^2$$

a) When the rod is subjected to axial pull (tension)

$$\text{Increase in length, } \delta l_1 = \frac{P_t l}{AE} = \frac{7000 \times 300}{615.75 \times 210 \times 10^3} = 0.016(+)$$

$$\text{Greatest length after elongation} = 300 + 0.016 = 300.016\ mm$$

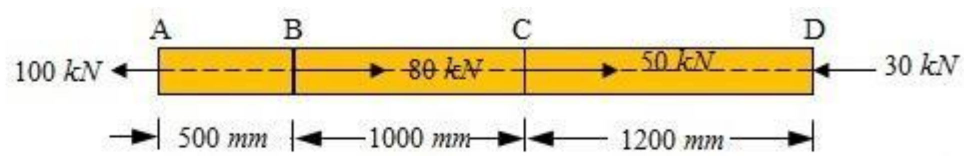
b) When the rod is subjected to axial push (compression)

$$\text{Increase in length, } \delta l_2 = \frac{P_c l}{AE} = \frac{16000 \times 300}{615.75 \times 210 \times 10^3} = 0.037(-)$$

$$\text{Least length after compression} = 300 - 0.037 = 299.963\ mm$$

$$\text{Difference between the greatest and least length} = 300.016 - 299.963 = 0.053\ mm$$

8. A straight bar of brass having cross-sectional area of 500 mm^2 is subjected to axial forces as shown in the figure.

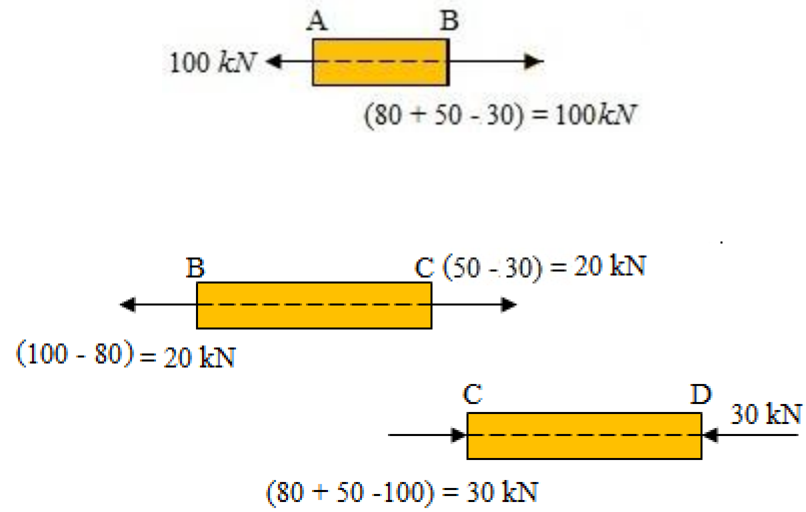


Find the total elongation of the bar. Take E = 80 *GPa*.

Solution. Given: Cross-sectional area, $A = 500\ mm^2$, $E = 80\ GPa = 80 \times 10^3\ N/mm^2$

For the sake of simplicity, the bar may be considered to be comprised of thee portions, AB, BC and CD. The elongation or contractions of each of the three portions are computed separately.

Equilibrium of each portion of the bar is considered for computation of the elongation or contraction.



Portion AB: AB is subjected to a tensile force of 100 kN

$$\text{Elongation of } AB, \delta_{AB} = \frac{P_1 l_1}{AE} = \frac{100 \times 1000 \times 500}{500 \times 80 \times 1000} = 1.25\, mm(+)$$

Portion BC: BC is subjected to a tensile force of 20 kN

$$\text{Elongation of } BC, \delta_{BC} = \frac{P_2 l_2}{AE} = \frac{20 \times 1000 \times 1000}{500 \times 80 \times 1000} = 0.5\, mm(+)$$

Portion BC: BC is subjected to a tensile force of 20 kN

$$\text{Contraction of } CD, \delta_{CD} = \frac{P_3 l_3}{AE} = \frac{30 \times 1000 \times 1200}{500 \times 80 \times 1000} = 0.9\, mm(-)$$

$$\text{Total elongation}, \delta = \delta_{AB} + \delta_{BC} + \delta_{CD} = 1.25 + 0.5 - 0.9 = 0.85\, mm$$

Complex stresses

In real life, except for a few simple cases, the structural components and machine parts are not as simple, that they would be subjected only to one dimensional (uniaxial force) stress. Instead, components of machineries and structural elements of large and complex civil engineering structures are most likely to be subjected to complex three dimensional stress systems due to the system of forces acting on them. In such situations, the analysis and failure of the structural elements involve analysis of complex stresses.

Even when subjected to uniaxial stress, instead of failure at the plane, normal to the force, the component may fail due to yielding at a different plane due to induced shear stress exceeding the permissible shear stress of the material of the member.

Hence, it is not always the case that the plane normal or parallel to the force would experience the maximum stress. The element under direct tension or compression will experience shear stress and that under shear stress will experience normal stress albeit at different planes.

To obtain a complete picture of the stresses in a bar, we must consider the stresses acting on an "inclined" (as opposed to a "normal") section through the bar.

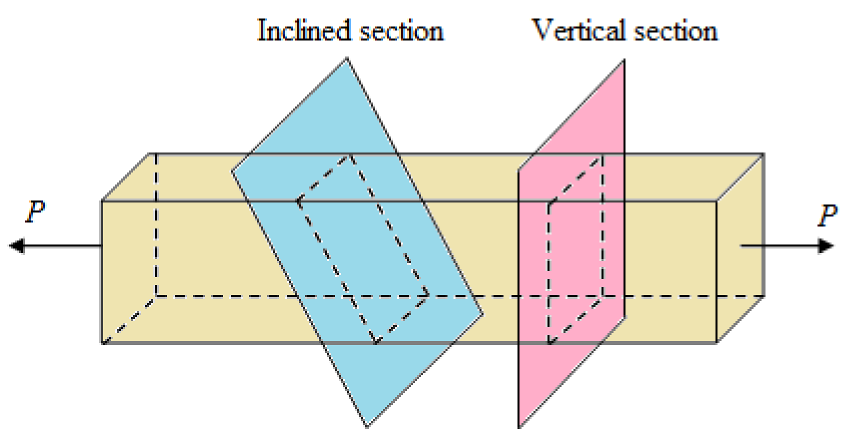


Fig. 1

Because the stresses are the same throughout the entire bar, the stresses on the sections are uniformly distributed.

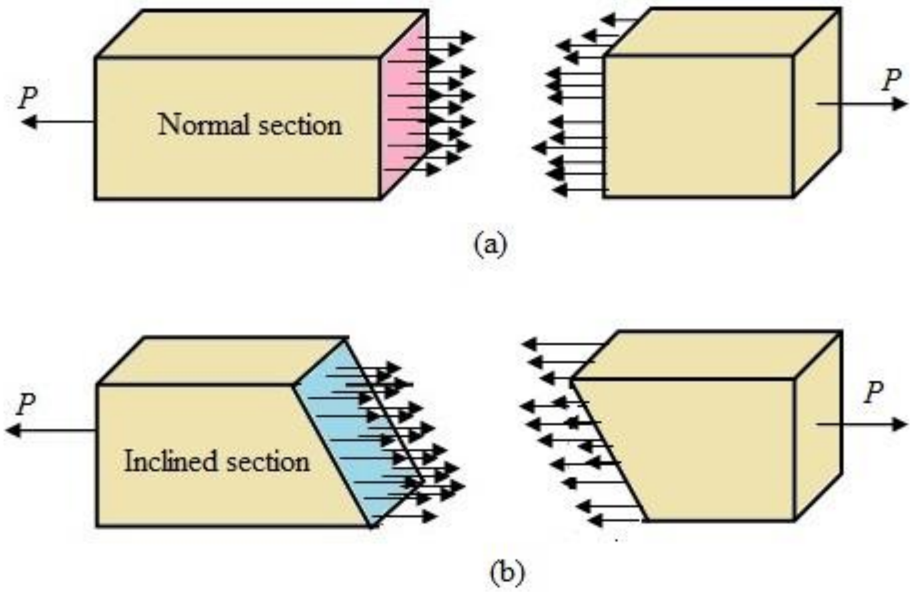


Fig. 2

Stresses in an oblique plane due to uniaxial normal stress:

In the previous section, we were computing the stresses on a plane when the force was either normal to the plane or tangential to it. In other words, the plane of interest was either normal to the force or parallel to it.

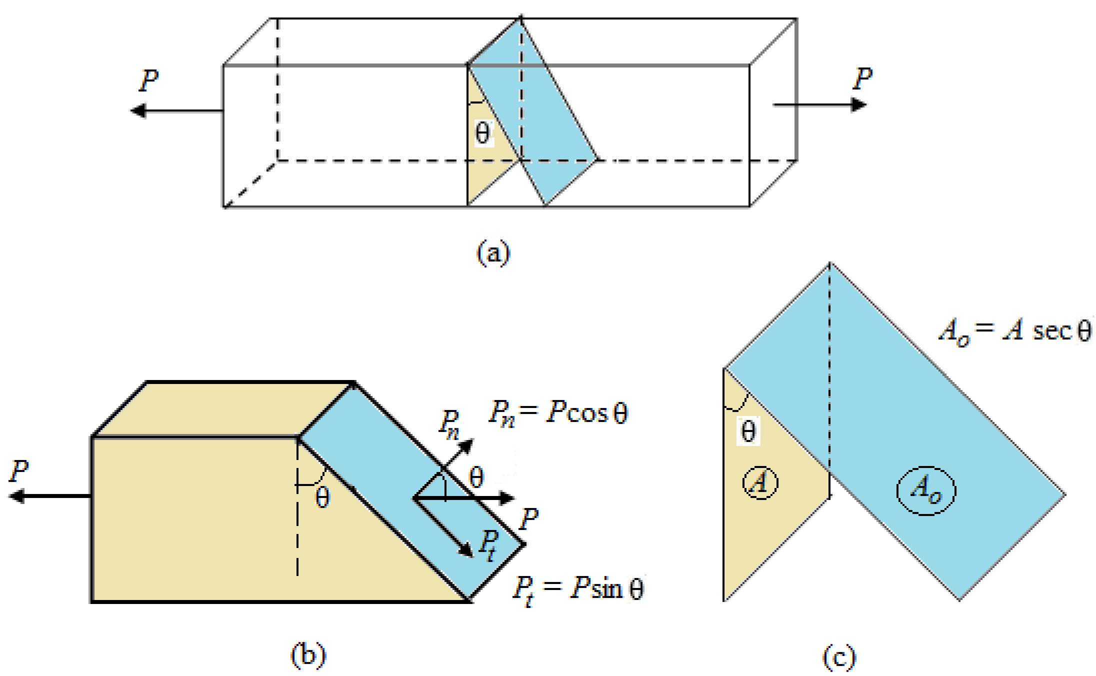


Fig. 3

In this section, the stresses on an oblique plane, i.e. plane making some angle (θ) with the vertical plane, will be dealt with. The normal stress on the vertical plane (perpendicular to the force) is $\sigma = P/A$, where P is the force and A is the area of cross-section of the plane.

In figure (b), however, there are two components of the force, one normal to the oblique plane (P_n) and the other tangential to the plane (P_t). Hence, the oblique plane will experience two stresses, one normal stress (σ_n) and the other tangential stress or shear stress (τ_n). The stress components can be calculated by dividing the respective forces with the area of the oblique plane.

Normal force perpendicular to the oblique plane, $P_n = P\cos\theta$

Shear force tangential to the oblique plane, $P_t = P\sin\theta$

Area of the oblique plane, $A_o = \dfrac{A}{\cos\theta} = A\sec\theta$

Normal stress, $\sigma_\theta = \dfrac{F_n}{A_o} = \dfrac{P\cos\theta}{A\sec\theta} = \dfrac{P\cos\theta.\cos\theta}{A}$

$$= \sigma\cos^2\theta \qquad (1)$$

Tangential or shear stress, $\tau_\theta = \dfrac{F_t}{A_o} = \dfrac{P\sin\theta}{A\sec\theta} = \dfrac{P\sin\theta\cos\theta}{A}$

$$= \frac{\sigma}{2}\sin 2\theta \qquad (2)$$

Thus, an oblique plane in a member under axial force will have two components of stresses on it, one normal to the plane while the other is tangential.

The stresses on the inclined plane, therefore, are not simply the resolutions of σ, perpendicular and tangential to that plane. The direct stress, σ_θ has a maximum value of σ, when $\theta = 0^0$ whilst the shear stress τ, has a maximum value of $\sigma/2$, when $\theta = 45^0$.

Thus, a material whose yield stress in shear is less than half that in tension or compression will yield initially in shear under the action of direct tensile or compressive forces.

Stresses in an oblique plane under pure shear stress system:

Consider a rectangular stressed element shown in Fig. 4 to which shear stresses (τ_{xy}) have been applied on the vertical sides so as to produce counterclockwise rotation. *Complementary shear stresses* of equal magnitude (τ_{yx}) but of opposite sense are then set up on the horizontal sides in order to prevent rotation of the element and to keep it in equilibrium.

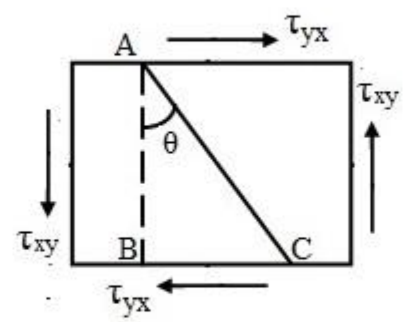


Fig. 4 Pure shear system

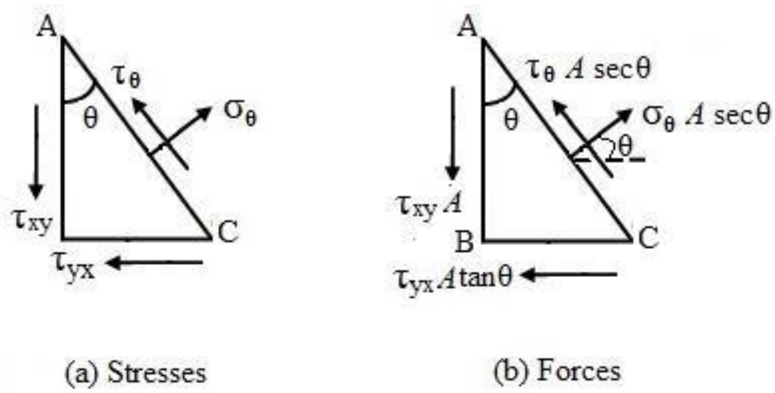


Fig. 5 Free body diagram of triangular prism

Consider the equilibrium of the triangular prism element in Fig. 5.
Resolving the forces along the direction normal to the plane AC, we have

$$\sigma_\theta \,.A\sec\theta = \tau_{xy} A\sin\theta + \tau_{yx} A\tan\theta\,\cos\theta$$

$$\sigma_\theta = \tau_{xy}\sin\theta\cos\theta + \tau_{yx}\sin\theta\cos\theta$$

$$= 2\tau_{xy} \sin\theta \cos\theta$$

$$= \tau_{xy} \sin 2\theta \qquad (3)$$

The maximum value of σ is τ_{xy} when $\theta = 45^0$.

Resolving the forces along the plane AC, we have

$$\tau_\theta . A\sec\theta + \tau_{yx} A \tan\theta \sin\theta = \tau_{xy} A \cos\theta$$

$$\tau_\theta .\sec\theta = \tau_{xy} \cos\theta - \tau_{yx} \tan\theta \sin\theta$$

$$\tau_\theta = \tau_{xy} \cos^2\theta - \tau_{yx} \sin^2\theta$$

$$= \tau_{xy} \left(\cos^2\theta - \sin^2\theta\right)$$

$$= \tau_{xy} \cos 2\theta \qquad (4)$$

The maximum value of τ_θ, is τ_{xy} when $\theta = 0^0$ or 90^0 and it has a value of zero when $\theta = 45^0$, i.e. on the planes of maximum direct stress.

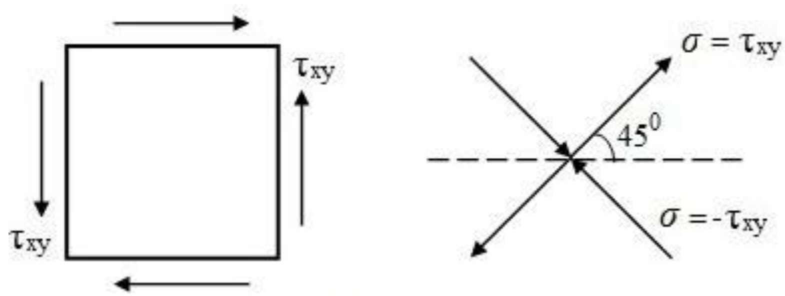


Fig. 6

Further consideration of eqn. (3) shows that the system of pure shear stresses produces an equivalent direct stress system as shown in Fig., one set compressive and one tensile, each at 45^0 to the original shear directions, and equal in magnitude to the applied shear.

Stresses in an oblique plane under biaxial normal stress system:

Consider the rectangular element as shown in Fig. 7 subjected to a system of two direct stresses, both tensile, at right angles, σ_x and σ_y.

For equilibrium of the portion *ABC,* resolving the forces perpendicular to *AC,*

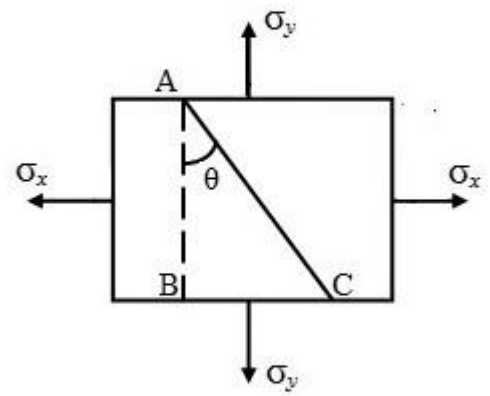


Fig. 7 Two -dimensioal normal stress system

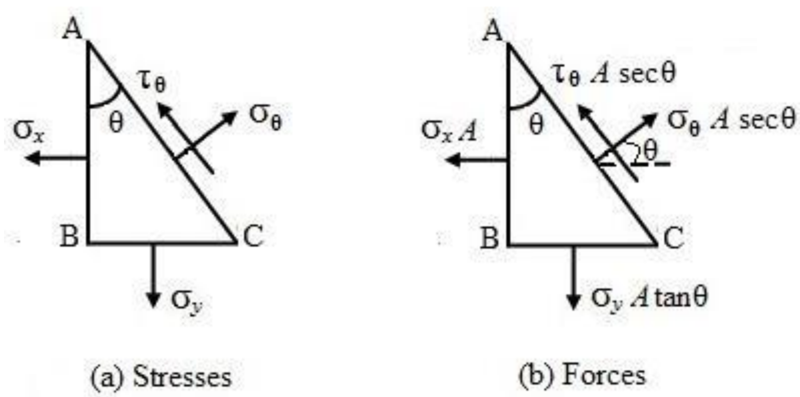


Fig. 8 Free body diagramof triangular prism

Consider the equilibrium of the triangular prism element in Fig. 8.

Resolving the forces along the direction normal to the plane AC, we have

$$\sigma_\theta .A\sec\theta = \sigma_x A.\cos\theta + \sigma_y A\tan\theta.\sin\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

$$= \frac{\sigma_x}{2}(1+\cos 2\theta) + \frac{\sigma_y}{2}(1-\cos 2\theta)$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad (5)$$

Resolving the forces along the plane AC, we have

$$\tau_\theta .A\sec\theta + \sigma_x A\sin\theta = \sigma_y A\tan\theta.\cos\theta$$

$$\tau_\theta . \sec\theta = -\sigma_x \sin\theta + \sigma_y \tan\theta . \cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta \cos\theta + \sigma_y \sin\theta . \cos\theta$$

$$= -(\sigma_x - \sigma_y) \sin\theta \cos\theta$$

$$= -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta \qquad (6)$$

The maximum direct stress will equal σ_x or σ_y, whichever is the greater, when $\theta = 0$ or 90^0. The maximum shear stress occurs in *the plane* when $\theta = 45^0$.

$$\tau_{max} = -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta$$

Stresses in an oblique plane under general two-dimensional stress system:

Consider a rectangular prism of uniform cross-sectional area under bi-axial/ two-dimensional stress as shown in the Fig 9.

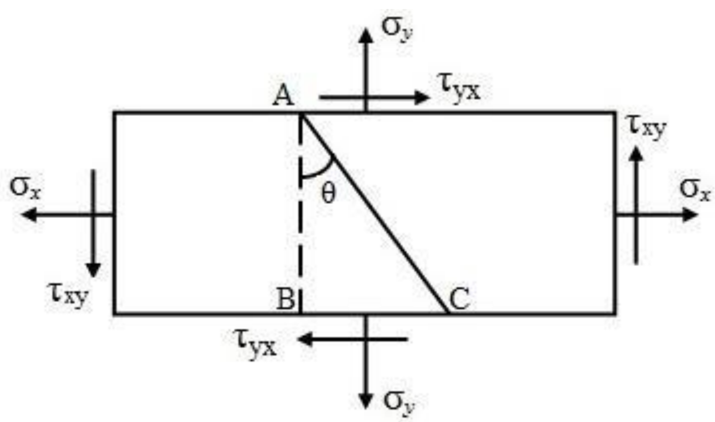


Fig. 9 Rectangular block under two-dimensional stress

Sign convention:

1. Angle of obliquity is measured in counter-clockwise direction with respect to vertical plane; the reference plane is taken as positive.
2. Tensile stress is taken as positive and compressive stress as negative.
3. Shear stress on vertical reference plane producing anticlockwise rotation is taken as positive.

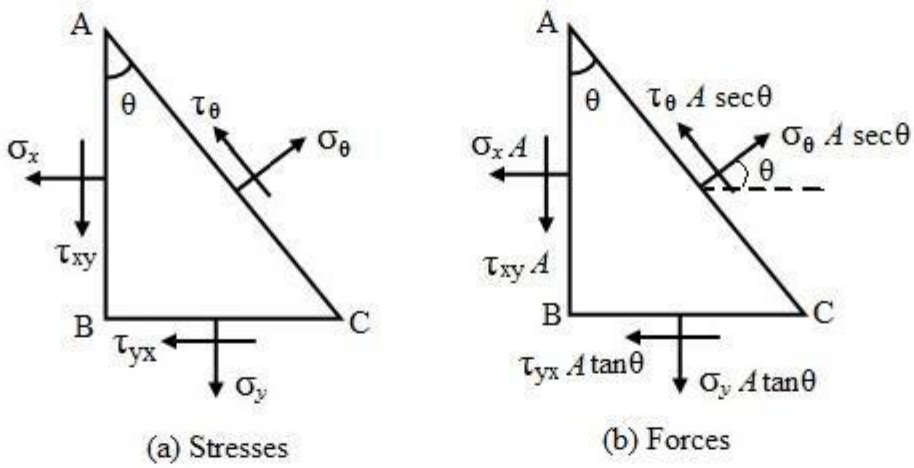


Fig.10 Free body diagram of triangular prism

Consider the equilibrium of the triangular prism element in Fig. 10.

Resolving the forces along the direction normal to the plane AC, we have

$$\sigma_\theta .A\sec\theta = \sigma_x A.\cos\theta + \tau_{xy} A\sin\theta + \sigma_y A\tan\theta.\sin\theta + \tau_{yx} A\tan\theta\cos\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin\theta\cos\theta + \tau_{yx}\sin\theta\cos\theta$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= \frac{\sigma_x}{2}(1+\cos 2\theta) + \frac{\sigma_y}{2}(1-\cos 2\theta) + \tau_{xy}\sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (7)$$

Resolving the forces along the plane AC, we have

$$\tau_\theta .A\sec\theta + \sigma_x A\sin\theta + \tau_{yx} A\tan\theta\sin\theta = \sigma_y A\tan\theta.\cos\theta + \tau_{xy} A\cos\theta$$

$$\tau_\theta .\sec\theta = -\sigma_x \sin\theta - \tau_{yx}\tan\theta\sin\theta + \sigma_y \tan\theta.\cos\theta + \tau_{xy}\cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta - \tau_{yx}\sin^2\theta + \sigma_y \sin\theta.\cos\theta + \tau_{xy}\cos^2\theta$$

$$= -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (8)$$

The maximum and minimum normal stresses (σ_1 and σ_2) are known as the **principal stresses**. The *maximum* and *minimum* normal stresses (σ_1 and σ_2) which occur on any plane in the member can now be determined as follows:

For σ_θ to be a maximum or minimum, $\frac{d\sigma_\theta}{d\theta} = 0$

We have $$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Since we are dealing with maximum and minimum normal stresses (principal stress), let us denote this angle as θ_p. The above equation can be written as,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \qquad (9)$$

There are two values of $2\theta_p$ in the range 0-360°, with values differing by 180°. There are two values of θ_p in the range 0-180°, with values differing by 90°. So, the planes on which the maximum and minimum normal stresses act are mutually perpendicular and are known as ***principal planes***.

We can now solve for the principal stresses by substituting for θ_p in the normal stress equation for σ_θ.

From triangle I and II of Fig. 11,

$$\sin 2\theta_p = \pm\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta_p = \pm\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

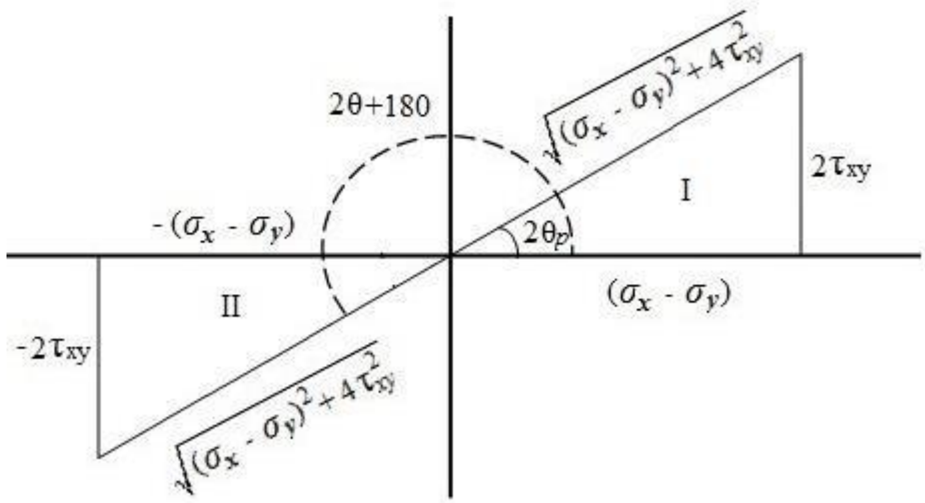


Fig. 11

Substituting the values of θ_p in (1), the maximum and minimum normal stresses are given by

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{(\sigma_x - \sigma_y)}{2} \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy} \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \qquad (10)$$

We have derived the maximum and minimum values of the normal stresses denoted as σ_1 (maximum) and σ_2 (minimum).

To find out which principal stress is associated with which principal angle, we could use the equations for sin θ_p and cos θ_p or for σ_θ.

Similarly substituting the value of θ_p in eqn. (8),

$$\tau_{\theta_p} = \mp \frac{(\sigma_x - \sigma_y)}{2} \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy} \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\tau_{\theta_p} = 0$$

Hence, the shear stresses are zero on the principal planes.

Now let us find out the maximum value of the shear stress. Differentiate eqn. (8) with respect to θ and equate it to zero. We get,

$$\frac{d\tau_\theta}{d\theta} = -\frac{(\sigma_x - \sigma_y)}{2}.2\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tau_{xy}\sin 2\theta = -\frac{(\sigma_x - \sigma_y)}{2}.\cos 2\theta$$

Since we are dealing with shear, let us denote this angle as θ_s. The above equation can be written as,

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} \qquad (11)$$

There are two values of $2\theta_s$ in the range 0-360°, with values differing by 180°. There are two values of θ_s in the range 0-180°, with values differing by 90°. So, the planes on which the maximum shear stresses act are mutually perpendicular.

Because shear stresses on perpendicular planes have equal magnitudes, the maximum positive and negative shear stresses differ only in sign.

Comparing the equation (4) for θ_s with that the equation (5) for θ_p, it is observed that both are reciprocal of each other. So we can write,

$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p} = -\cot 2\theta_p$$

$$\tan 2\theta_s = \tan(90 + 2\theta_p)$$

$$\theta_s = 45 + \theta_p$$

So, the planes of maximum shear stress (θ_s) occur at 45° to the principal planes (θ_p). We have therefore derived maximum & minimum values of principal stresses, their angles, maximum values of shear stress and its orientation with respect to principal planes.

Further, from triangle I and II of Fig. 12,

$$\sin 2\theta_p = \mp \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta_p = \pm \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

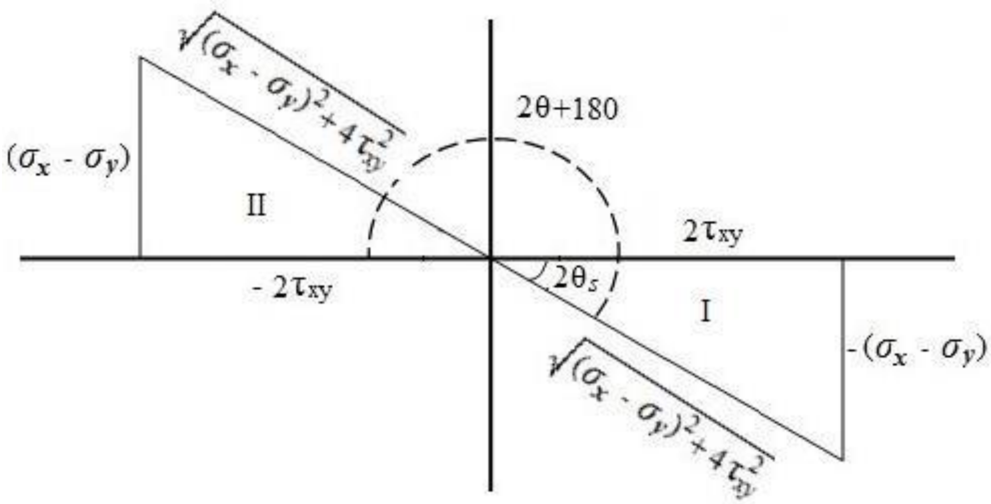


Fig. 12

Substituting the values of θ_s in (2), the maximum and minimum shear stresses are given by

$$\tau_{max} = \pm \frac{(\sigma_x - \sigma_y)}{2} \cdot \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy} \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\tau_{max} = \pm . \frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} = \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$\tau_{max} = \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

In a structural member under biaxial complex stress system, there exist two mutually perpendicular planes on which the normal stress is either maximum or minimum. These planes are known as ***principal planes***. The shear stress on these planes is zero.

A little observation will show that, $\tau_{max} = \pm \frac{1}{2}(\sigma_1 - \sigma_2)$, where σ_1 and σ_2 are principal stresses.

Further, it can be seen that the sum of the principal stresses is same as the sum of the normal stresses in any two mutually perpendicular directions.

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

Transformation of stress coordinates:

There exists only one intrinsic/fundamental state of stress at a point in a stressed body akin to the position of a point on a plane. As the coordinates of a point (*x, y*) on a plane changes with the orientation of coordinate axes of reference, so does the stress coordinates (σ, τ) of a point in a stressed body (with respect to the plane of reference) with the orientation of its plane of consideration.

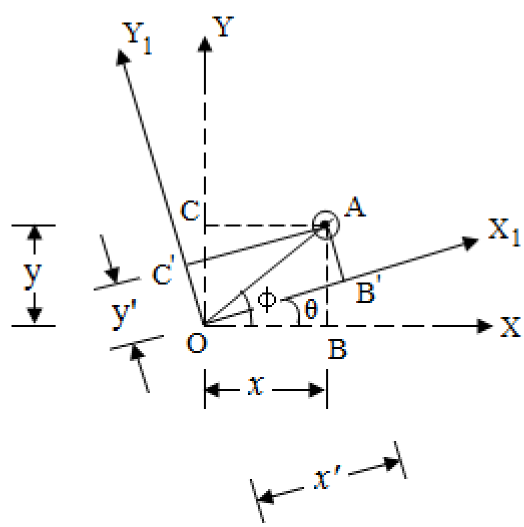


Fig. 13

'*A*' is a point on a plane with coordinates axes *OX* and *OY* as shown in Fig. 13. The coordinates of *A* is (*x, y*). If the orientation of axes of reference *OX* and *OY* is changed by an angle of θ in counterclockwise direction, then the position is not going to be changed. Instead its coordinates (x_1, y_1) in respect of changed coordinate axes will change.

$d =$ disance of A from the origin O of the plane

$\phi =$ angular distance of A from the origin O of the plane

$$x = d\cos\phi,\ y = d\sin\phi$$

$$x_1 = d\cos(\phi - \theta), y_1 = d\sin(\phi - \theta)$$

It is evident that only the coordinates of the point and not its inherent position depends on the orientation of the axes of reference. Similarly, regardless of the orientation of the element used to portray the state of stress, the intrinsic state of stress remains the same. In other words the stress coordinates (σ, τ) and not the inherent state of stress of a point change.

Member subjected to principal stresses:

Mohr's circle:

Mohr's circle is a graphical representation of stress transformation equations. The equations of stress transformation describe a circle if normal stress and shear stress are represented as abscissa and ordinate respectively. Each point on the circumference of Mohr's circle represents a plane through the centre of the circle and the coordinates (σ, τ) of the point represents the normal stress (σ) and shear stress (τ) on the given plane.

Mohr's circle can be drawn from a given state of stress at a point in a structural member. Consider a stress element representing the state of stress at a point as shown in the Fig. 14.

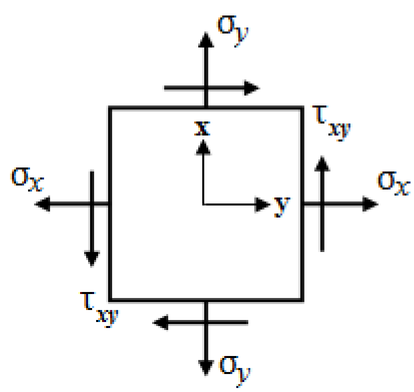


Fig. 14

Stress transformation equations for normal and tangential components on a plane are given by

Normal stress on the plane, $\sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$ (12)

Shear stress on the plane, $\tau_\theta = -\dfrac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$ (13)

Rearranging the equation (12), we have

$$\sigma_\theta - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (14)$$

Squaring both sides of equation (13) and (14) and adding them together, we have

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left[\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right]^2$$

This is the equation of a circle with centre $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ and radius $R = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ and this circle is known as Mohr's circle named after the German Civil Engineer Otto Mohr (1835-1918). It provides a simple and clear picture of an otherwise complicated analysis.

Procedure for drawing Mohr's circle:

1. Draw coordinates axes in Cartesian coordinate system with *O* as origin, normal stress (σ) as abscissa (positive to the right) and shear stress (τ) as ordinate (positive downward).
2. Locate the centre C of the circle at the point having coordinates $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$.
3. Locate point *A*, representing the state of stress on the vertical plane, i.e., face *x* of the element by plotting its coordinates σ_x and τ. Point *A* on the circle corresponds to $\theta = 0^0$ and represents the vertical plane.
4. Locate point *B*, representing the state of stress on the horizontal plane, i.e., face y of the element by plotting its coordinates σ_y and $-\tau$. Point *A* on the circle corresponds to $\theta = 90^0$ and represents the horizontal plane.
5. Join *AB* so as to intersect the normal stress axis at *C*.
6. With the point *C* as the centre and *CA* (= *CB*) as radius, draw Mohr's circle through points *A* and *B*. This is the required Mohr's circle which has radius *R*.

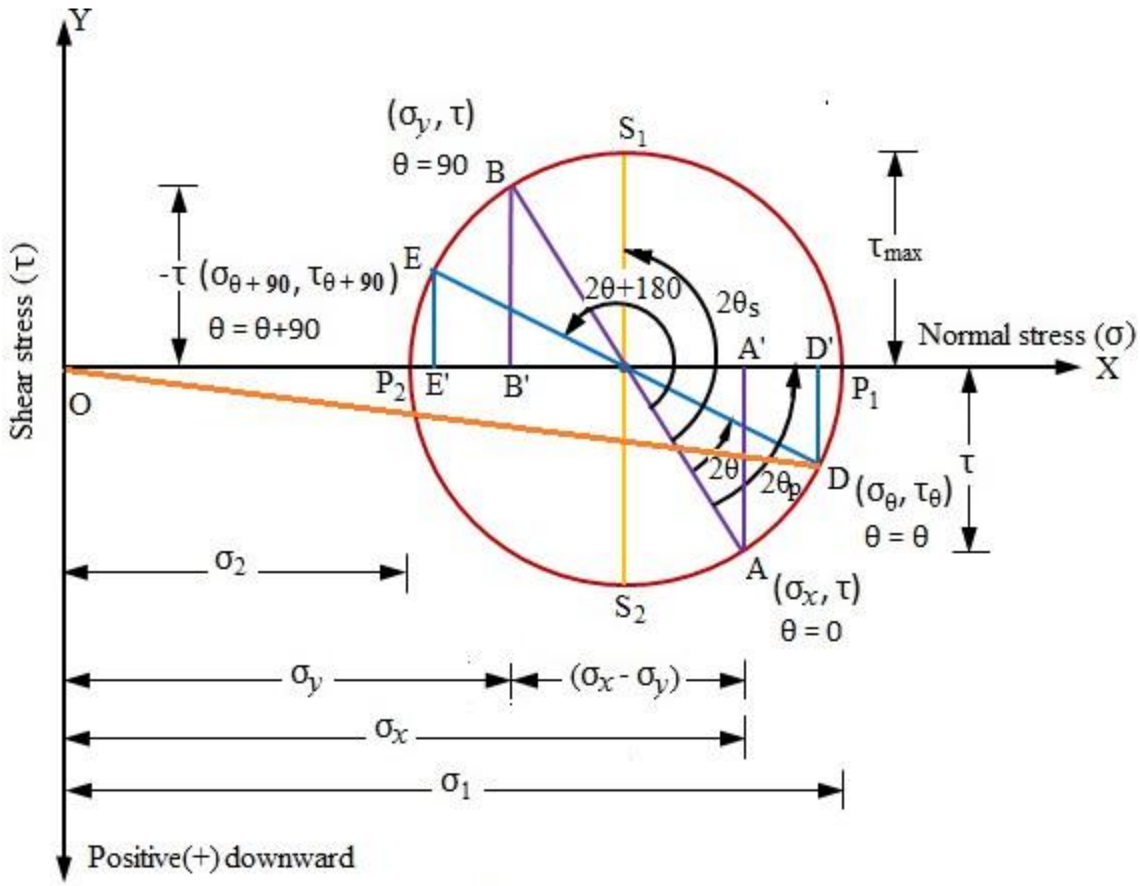


Fig. 15 Mohr's circle

Every point on the circumference of the circle then represents a state of stress on some plane through *C*.

The stress state on an inclined plane with an angle θ is represented at point *D* on the Mohr's circle, which is measured an angle 2θ counter- clockwise from point *A* to show the coordinate at *D*.

Consider any point *D* on the circumference of the circle, such that *CD* makes an angle 2θ with *CA*, and drop a perpendicular from *D* to meet the σ axis at *D'*.

Coordinates of *D*:

$$OD' = OC + CD' = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + R\cos\left(2\theta_p - 2\theta\right)$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + R\cos 2\theta_p \cos 2\theta + R\sin 2\theta_p \sin 2\theta$$

But $$R\cos 2\theta_p = \frac{1}{2}(\sigma_x - \sigma_y) \textit{ and } R\sin 2\theta_p = \tau_{xy}$$

Therefore, $$OD' = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin 2\theta$$

On inspection this is seen to be eqn. (12) for the normal stress σ_θ on the plane inclined at θ to the vertical plane *AB*.

Similarly, $$DD' = R\sin(2\theta_p - 2\theta)$$

$$DD' = R\sin 2\theta_p \cos 2\theta - R\cos 2\theta_p \sin 2\theta$$

$$DD' = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy} \cos 2\theta$$

Again, on inspection this is seen to be eqn. (13) for the shear stress τ_θ on the plane inclined at θ to the vertical plane *AB*.

Thus the coordinates of ***Q*** are the normal and shear stresses on a plane inclined at θ to *AB* in the original stress system.

Characteristics of Mohr's circle

1. The direct stress is maximum when *D* is at P_1, i.e. OP_1 is the length representing the maximum principal stress σ_1 and $2\theta_p$ gives the angle of the plane θ_p, from *AB*. Similarly, OP_2 is the other principal stress.
2. The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle. This follows since shear stresses and complementary shear stresses have the same value; *therefore the centre of the circle will always lie on the* σ-*axis midway between* σ_x and σ_y.

3. From the above point the direct stress on the plane of maximum shear must be midway between σ_x and σ_y, i.e. $\frac{1}{2}(\sigma_x + \sigma_y)$.
4. The shear stress on the principal planes is zero.
5. Since the resultant of two stresses at 90^0 can be found from the parallelogram of vectors as the diagonal, as shown in Fig. **13.10,** the resultant stress on the plane at θ to *AB* is given by *OD* on Mohr's circle.
6. Comparing the Mohr's circle and the stress element, it is observed points S_1 and S_2 representing the points of maximum and minimum shear stresses, are located on the circle at 90^0 from points P_1 and P_2 i.e. the planes of maximum and minimum shear stress are at 45^0 to the principal planes, and

The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction. It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.

Numerical

1. A material has permissible stresses in tension, compression and shear as 30 *N/mm*^{2}, 90 *N/mm*^{2} and 25 *N/mm*^{2} respectively. If specimens of diameter 20 *mm* are tested in tension and compression, identify the failure surfaces and failure load.

Solution:

Case I (Test in tension): If subjected to full tensile strength,

Maximum tensile stress, $\sigma_t = 30\, N/mm^2$

Corresponding maximum stress in shear, $\tau_{\max} = \frac{\sigma_t}{2} = \frac{30}{2} = 15\, N/mm^2 < 25\, N/mm^2$

Hence, the failure will occur due to tension. The maximum tensile stress is in the axial direction. Hence failure will occur on the plane of axial tensile stress, i.e., at right angle to the stress.

Corresponding tensile force, $P_t = \text{Area of specimen} \times \text{tensile strength}$

$$= A \times \sigma_t = \frac{\pi}{4} \times 20^2 \times 30$$

$$= 9224.778\, N$$

Case II (Test in compression):

Maximum compressive stress, $\sigma_c = 90\, N/mm^2$

Corresponding maximum stress in shear, $\tau_{\max} = \frac{\sigma_c}{2} = \frac{90}{2} = 45\, N/mm^2 > 25\, N/mm^2$

Hence, the failure will occur due to shear. The failure will occur on the plane at 45 to the plane of axial stress.

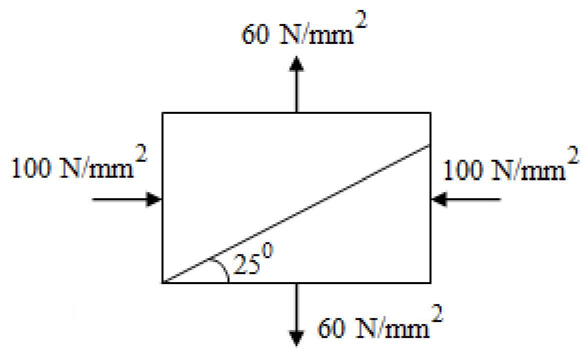
Corresponding compressive stress causing shear failure, $\sigma_{cf} = 2 \times 25 = 50\, N/mm^2$

Corresponding compressive force, $P_c = \text{Area of specimen} \times \text{compressiv estress}$

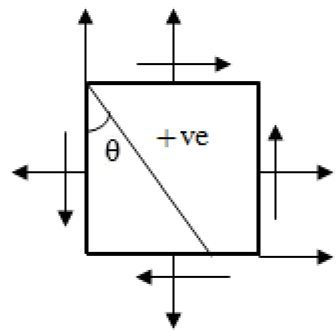
$$= A \times \sigma_{cf} = \frac{\pi}{4} \times 20^2 \times 50$$

$$= 15707.93\, N$$

2. Normal stresses acting at a point in a strained material are 100 *N/mm*^{2} compressive and 60 *N/mm*^{2} tensile as shown in the figure. Find the stresses on the given oblique plane.



Solution: Sign convention:

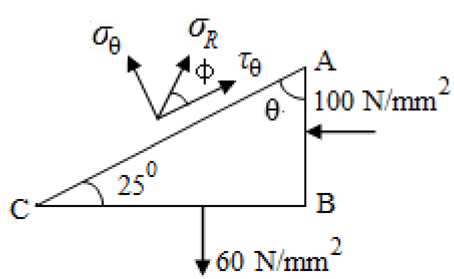


Sign convention

Define the stresses in terms of the sign convention:

$\sigma_x = -100\,MPa,\ \sigma_y = 60\,MPa$

Angle of orientation of plane of plane AC, θ = - 65^0 (clockwise)



Normal stress on the oblique plane, $\sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta$

$$= \frac{-100+60}{2} + \frac{-100-60}{2}\cos 2\left(-65^0\right)$$

$$= -20 - 80\cos\left(-130^0\right)$$

$$= 31.423\ N/mm^2\ \text{(tensile)}$$

Shear stress on the oblique plane, $\tau_\theta = -\dfrac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta$

$$= -\frac{(-100-60)}{2}\sin 2(-65^0)$$

$$= 80\sin(-130^0)$$

$$= -61.284\, N/mm^2 \text{ (Clockwise)}$$

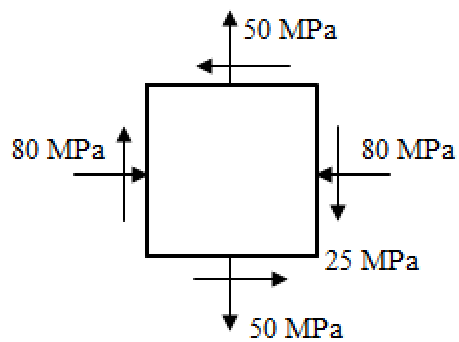
Resultant stress on the plane, $\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{31.432^2 + (-61.284)^2}$

$$= 68.87\, N/mm^2$$

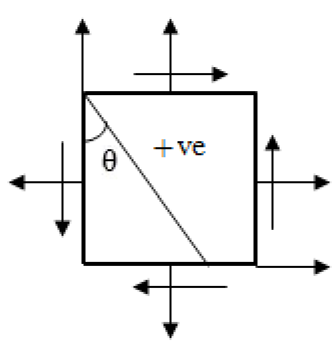
Angle of obliquity, $\phi = \tan^{-1}\left(\frac{\sigma_\theta}{\tau_\theta}\right) = \tan^{-1}\left(\frac{31.423}{61.284}\right)$

$$= 27.14^0$$

3. The state of plane stress at a point is represented by the stress element below. Determine the stresses acting on plane oriented 30° clockwise with respect to the vertical plane.



Solution: Sign convention:



Sign convention

Define the stresses in terms of the sign convention:

$$\sigma_x = -80\,MPa,\ \sigma_y = 50\,MPa \quad and\ \tau_{xy} = -25\,MPa$$

Angle of orientation of plane, $\theta = -30^0$

Normal stress on the given plane, $\sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

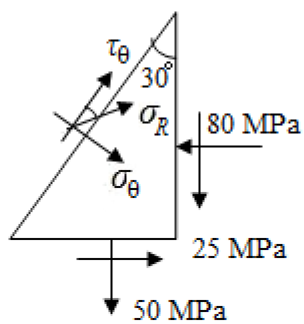
$$= \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(-30) + (-25)\sin 2(-30)$$

$$= \frac{-30}{2} + \frac{-130}{2}\cos(-60) - 25\sin(-60)$$

$$= -15 - 65\cos(-60) - 25\sin(-60)$$

$$= -15 - 65\cos(-60) - 25\sin(-60)$$

$$= -25.9\,MPa \quad \text{(Compressio n)}$$



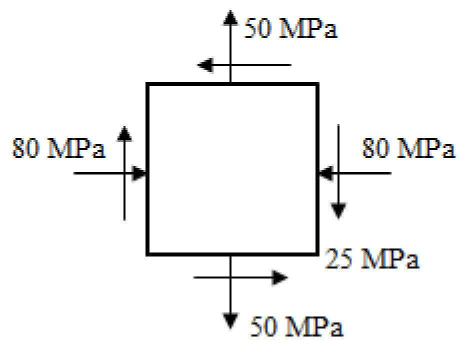
Shear stress on the oblique plane, $\tau_\theta = -\dfrac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$

$$= -\frac{(-80-50)}{2}\sin 2(-30^0) + (-25)\cos 2(-30^0)$$

$$= 65 \times \sin(-60^0) - 25\cos(-60^0)$$

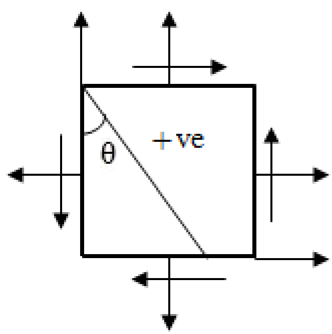
$$= -68.80\,MPa \quad \text{(Clockwise)}$$

4. Example: The state of plane stress at a point is represented by the stress element below. Determine the principal stresses and draw the corresponding stress element.



Solution:

Sign convention:



Sign convention

Define the stresses in terms of the sign convention:

$$\sigma_x = -80\,MPa,\ \sigma_y = 50\,MPa \quad and\ \tau_{xy} = -25\,MPa$$

Angle of orientation of plane, $\theta = -30^0$

Principal stresses, $\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$

$$= \frac{1}{2}(-80+50) \pm \frac{1}{2}\sqrt{(-80-50)^2 + 4(-25)^2}$$

$$= -15 \pm \frac{1}{2}\sqrt{(-130)^2 + 4 \times 625}$$

$$= -15 \pm 69.64$$

$$\sigma_1 = 54.64 MPa \ \ and\ \sigma_2 = -84.64 MPa$$

Angle of principal plane, $\tan 2\theta_p = \dfrac{2\tau_{xy}}{(\sigma_x - \sigma_y)} = \dfrac{2(-25)}{(-80-50)}$

$$\tan 2\theta_p = 0.3846$$

$$2\theta_p = 21.0^0 \text{ and } 21.0^0 + 180^0$$

$$\theta_p = 10.5^0 \text{ and } 100.5^0$$

Check for which angle goes with which principal stress. Put $\theta = 10.5^0$ in the stress equation,

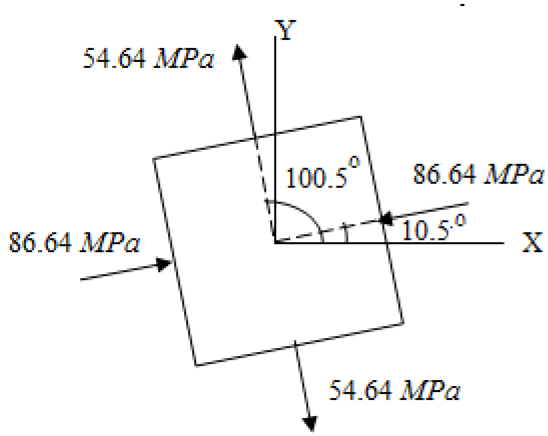
$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(10.5^0) + (-25)\sin 2(10.5^0)$$

$$= -84.6\ MPa$$

$$\sigma_1 = 54.64 MPa\ with\ \theta_{p1} = 100.5^0$$

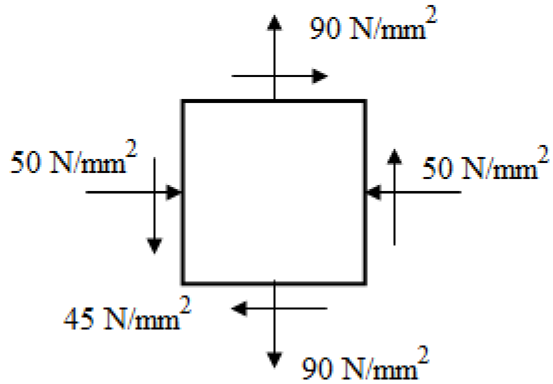
$$\sigma_2 = -84.64 MPa\ with\ \theta_p = 10.5^0$$



5. A point in a stressed structural member is subjected to a tensile stress of 90 N/mm^2 on a horizontal plane and compressive stress of 50 N/mm^2 on the vertical plane. There is a shear

stress of 45 N/mm^2 such that when on vertical plane, it tends to rotate the member in counterclockwise direction. Determine the maximum shear stress and also the resultant stress on the planes of maximum shear stresses.

Solution:



Define the stresses in terms of the sign convention:

$$\sigma_x = -50\,MPa,\ \sigma_y = 90\,MPa \quad and\ \tau_{xy} = 45\,MPa$$

Angle of orientation of plane, θ = - 30^0

Maximum shear stress, $\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$

$$= \frac{1}{2}\sqrt{(-50-90)^2 + 45^2} = 73.53\,N/mm^2\ (counterclockwise)$$

Inclination of plane of maximum shear to the vertical plane,

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} = -\frac{(-50-90)}{2\times 45}$$

$$\tan 2\theta_s = -1.555$$

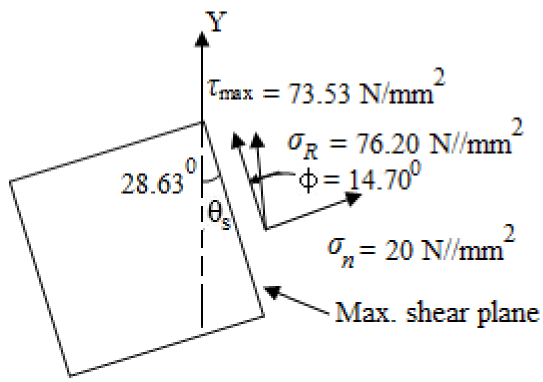
$$2\theta_s = \tan^{-1}(1.555) = 57.26^0 \quad and\ \ 237.26^0$$

$$\theta_s = 28.63^0\ \ and\ 118.63^0 \qquad (counterclockwise)$$

Normal stress on plane of maximum shear stress,

$$\sigma_n = \frac{-50+90}{2} + \frac{-50-90}{2}\cos 2\left(28.63^0\right) + 45\sin 2\left(28.63^0\right)$$

$$= 20\, N/mm^2$$



Resultant stress on the plane, $\sigma_R = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{20^2 + 73.53^2}$

$$= 76.20\, N/mm^2$$

$$\phi = \tan^{-1}\left(\frac{\sigma_n}{\tau_{max}}\right) = \tan^{-1}\left(\frac{20}{76.20}\right)$$

$$= 14.70^0$$

Summary

1. The normal stress **σ** and shear stress τ on oblique planes resulting from direct loading are

$$\sigma_\theta = \sigma \cos^2\theta$$

$$\tau_\theta = \frac{\sigma}{2}\sin 2\theta$$

2. The stresses on oblique planes owing to a complex stress system are:

Normal stress, $\sigma_\theta = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

Normal stress, $\tau_\theta = -\dfrac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$

3. The ***principal stresses*** (i.e. the maximum and minimum direct stresses) are then

$$\sigma_{1,2} = \frac{1}{2}\left(\sigma_x + \sigma_y\right) \pm \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

and these occur on planes at an angle θ_p to the vertical plane on which σ_x, acts, given by either

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\left(\sigma_x - \sigma_y\right)}$$

where $\sigma_p = \sigma_1$ or σ_1**,** the planes being termed ***principal planes.*** The principal planes are always at 90^0 to each other, and the ***planes*** of ***maximum shear*** are then located at 45^0 to them.

4. The ***maximum shear stress*** is

$$\tau_{max} = \pm\frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

$$\tau_{max} = \pm\frac{1}{2}\left(\sigma_1 - \sigma_2\right)$$

5. Normal stress on plane of maximum shear $= \frac{1}{2}\left(\sigma_x + \sigma_y\right)$
6. Shear stress on plane of maximum direct stress (principal plane) = 0
7. In problems where the principal stress in the third dimension σ_3 either is known or can be assumed to be zero, the true maximum shear stress is then

$$\frac{1}{2}\left(greatest principal\ stress - least\ principal\ stress\right)$$

Bending Stress

When a beam is loaded with external loads, all the sections of the beam will experience bending moments and shear forces. The shear forces and bending moments at various sections of the beam can be evaluated as discussed in the earlier chapter. In this chapter, the bending and bending stress distribution across a section will be dealt with.

Some practical applications of bending stress shall also be dealt with. These are

1. Moment carrying capacity of a section
2. Evaluation of extreme normal stresses due to bending
3. Design of beam for bending
4. Evaluation of load bearing capacity of the beam

The major stresses induced due to bending are normal stresses of tension and compression. But the state of stress within the beam includes *shear stresses due to the shear force* in addition to the major *normal stresses due to bending* although the former are generally of *smaller order* when compared to the latter.

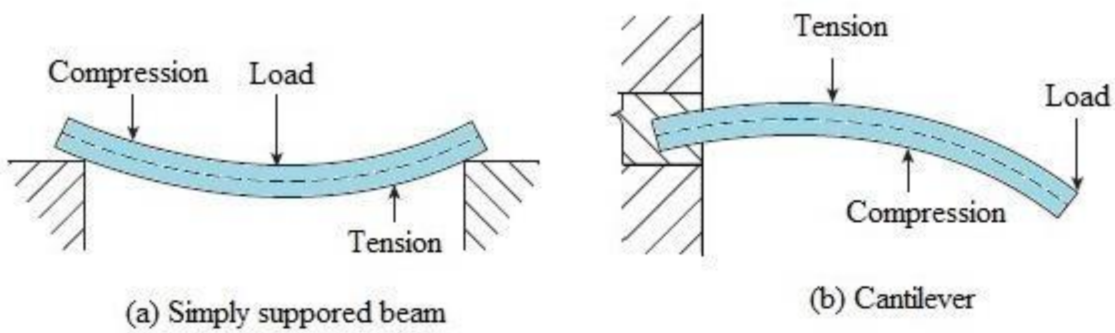


Fig. 1

Simple Bending or Pure Bending

A beam or a part of it is said to be in a state of pure bending when it bends under the action of uniform/constant bending moment, without any shear force.

Alternatively, a portion of a beam is said to be in a state of simple bending or pure bending when the shear force over that portion is zero. In that case there is no chance of shear stress in the beam. But, the stress that will propagate in the beam as a result will be known as normal stress.

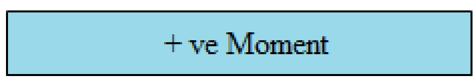
However, in practice, when a beam is subjected to transverse loads, the bending moment at a section is accompanied by shear force. But, it is generally observed that the shear force is zero where the bending moment is maximum. Therefore, the condition of pure bending or simple bending is deemed to be satisfied at that section.

Examples of pure bending are –

1. Bending of simple supported beam due to end coupling (Uniform pure bending)
2. Bending of cantilever beam with end moment (Uniform pure bending)
3. Bending of the portion between two equal point loads in a simple supported beam with two-point loading (Non-uniform pure bending)

The *four point bending* of the simply supported beam

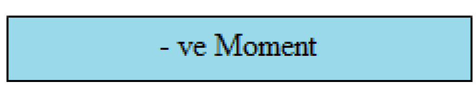
(a) Simple supported beam with end coupling



(b) Bending moment diagram

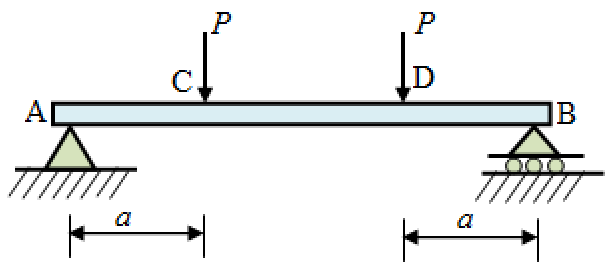
Fig. 2

(a) Cantilever subjected to moment at its end

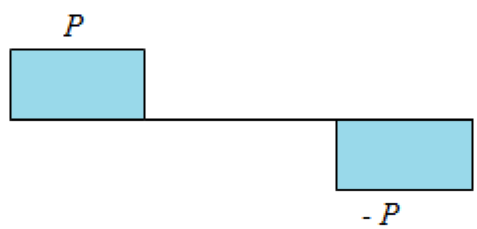


(b) Bending moment diagram

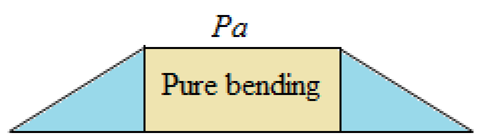
Fig. 3



(a) Simply supported beam with two-point loading



(b) Shear force diagram



(c) Bending moment diagram

Fig. 4

Theory of Simple Bending

The theory which deals with the determination of stresses at a section of a beam due to pure bending is called ***theory of simple bending***. In this chapter, bending of *straight homogeneous* beams of *uniform cross sectional area* with vertical *axis of symmetry* shall be considered. The application of this theory can be extended to beams with two or more different materials as well as curved beams.

Several cross-sections of beams satisfying the above conditions are shown in the Fig. 5.

A beam of rectangular cross-section with typical loading condition is shown in the Fig. 6. Also shown in the Fig. 7 is the three-dimensional beam with longitudinal vertical plane of symmetry, with the cross-section symmetric about this plane. It is assumed that the loading and supports are also symmetric about this plane. With these conditions, the beam has no tendency to twist and will undergo bending only.

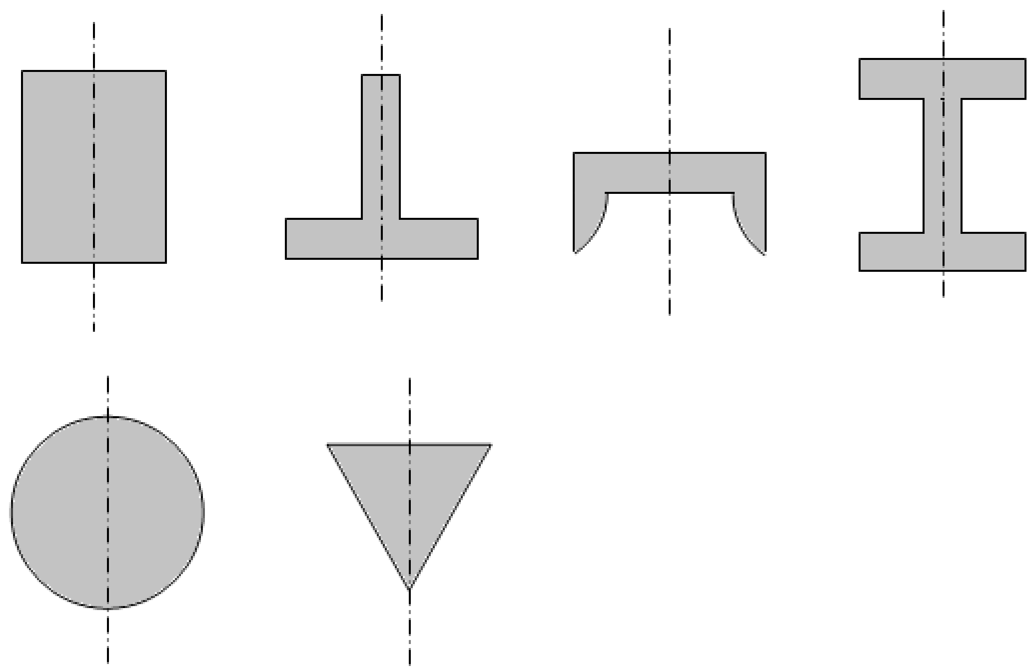


Fig. 5 Beam cross-sections with vertical axis of symmetry

A beam subjected to sagging moment is shown in the Fig. 8. The beam is imagined to be consisting of a number of longitudinal fibres; one such fibre is is shown in colour. It is obvious that the fibres near the upper side of the beam are compressed; hence an element in the upper part is under compression. The fibres at the bottom side of the beam get stretched and, hence, the elements on the lower side are subjected to tension. Somewhere in between, there will be a plane where the fibres are subjected to neither tension nor compression. Such a plane is termed as ***neutral surface*** *or* ***neutral plane***.

In the conventional coordinate system attached to the beam in Fig. 8, x axis is the longitudinal axis of the beam, the y axis is in the transverse direction and the longitudinal plane of symmetry is in the x- y plane, also called the ***plane of bending***.

Neutral Surface

The longitudinal surface of a beam under bending which experiences neither tension nor compression is known as neutral surface. There is only one neutral surface in a beam.

Neutral Axis

The line of intersection of transverse section of beam with the neutral surface is known as neutral axis. In other words, the line of intersection of the longitudinal plane of symmetry and the neutral surface is known as neutral axis. Neutral axis experiences no extension or contraction.

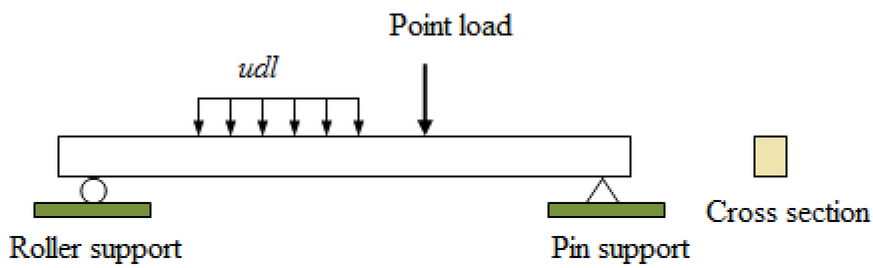


Fig. 6 Loaded beam

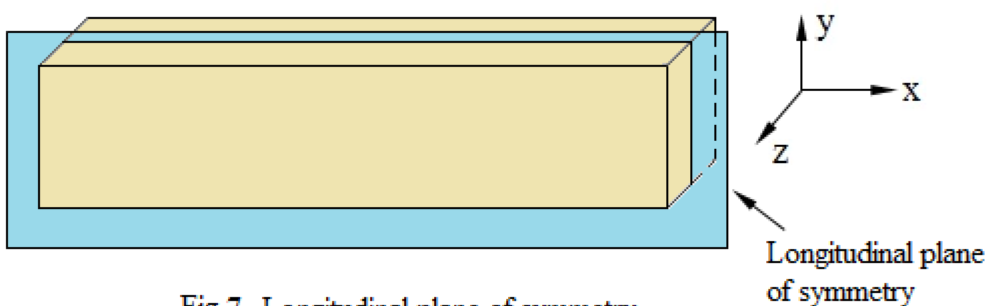


Fig. 7 Longitudinal plane of symmetry

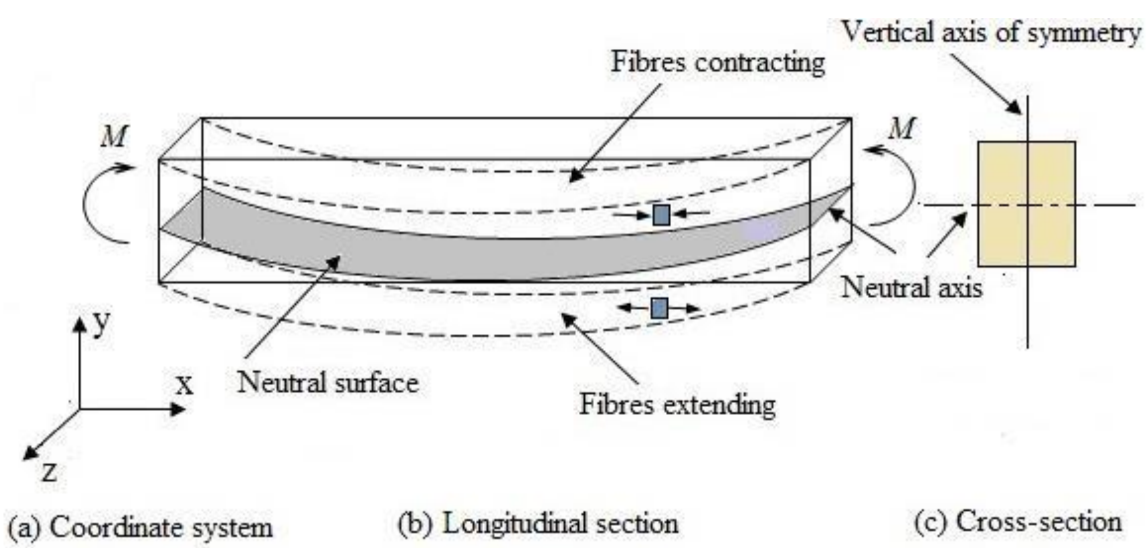


Fig. 8 Pure moment (sagging)

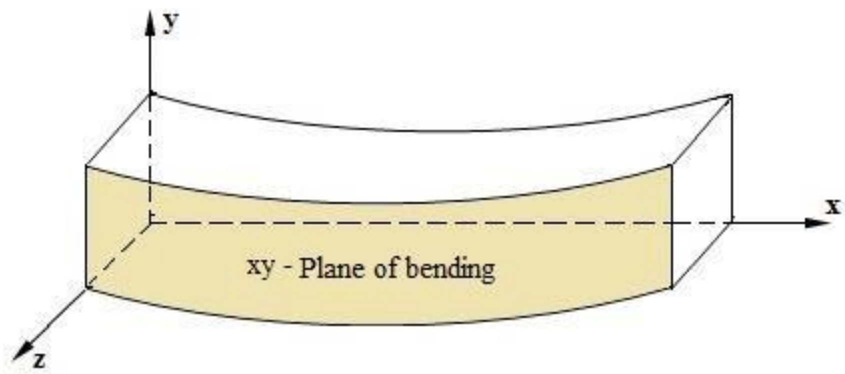


Fig. 9 Plane of bending

Axis of beam

The intersection of the longitudinal plane of symmetry and the neutral surface is called the **axis of the beam.** In other words, the line through the centroid of all the cross-sections of the beam is known as axis of the beam.

Assumptions for theory of pure bending:

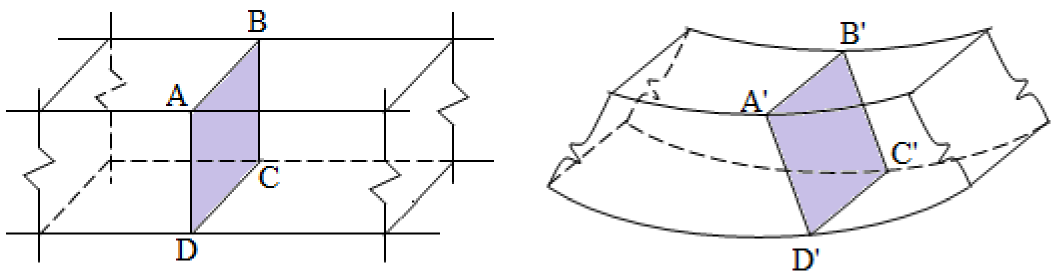
The assumptions made in the theory of simple bending are as follows:

1. The material of the beam is perfectly homogeneous (i.e. of the same kind throughout) and isotropic (i.e. of same elastic properties in all directions).
2. The material is stressed within elastic limit and obeys Hooke's law.
3. The value of modulus of elasticity for the material is same in tension and compression.
4. The beam is subjected to pure bending and therefore bends in the form of an arc of a circle.
5. The transverse sections, which are plane and normal to the longitudinal axis before bending, remain plane and normal to the longitudinal axis of the beam after bending.
6. The radius of curvature of the bent axis of the beam is large compared to the dimensions of the section of beam.
7. Each layer of the beam is free to expand or contract independently.
8. The cross-sectional area is symmetric about an axis perpendicular to the neutral axis.

Explanation of the assumptions

According to assumption No. 5, plane section *ABCD* before bending as shown in Fig. 10 remains plane after bending as shown by *A'B'C'D'*. This assumption, also known as Bernoulli's assumption, is perfectly valid for beams with pure bending. If there is any shear along with the bending, the shear deformation distorts the plane and *A'B'* will not remain plane. However, for beams with smaller depth (d<1/10th span) shear deformation is small and this assumption is not much affected. In case of deep beams, with shear forces, this assumption fails.

Assumption No. 6, the radius of curvature is large compared to depth is valid if deflections are less than 1/10th to 1/5th of depth of beam. Therefore, the theory derived with this assumption may be called ***small deflection theory***.



(a) Before bending (b) After bending

Fig. 10

Relationship between Bending Stress and Radius of Curvature

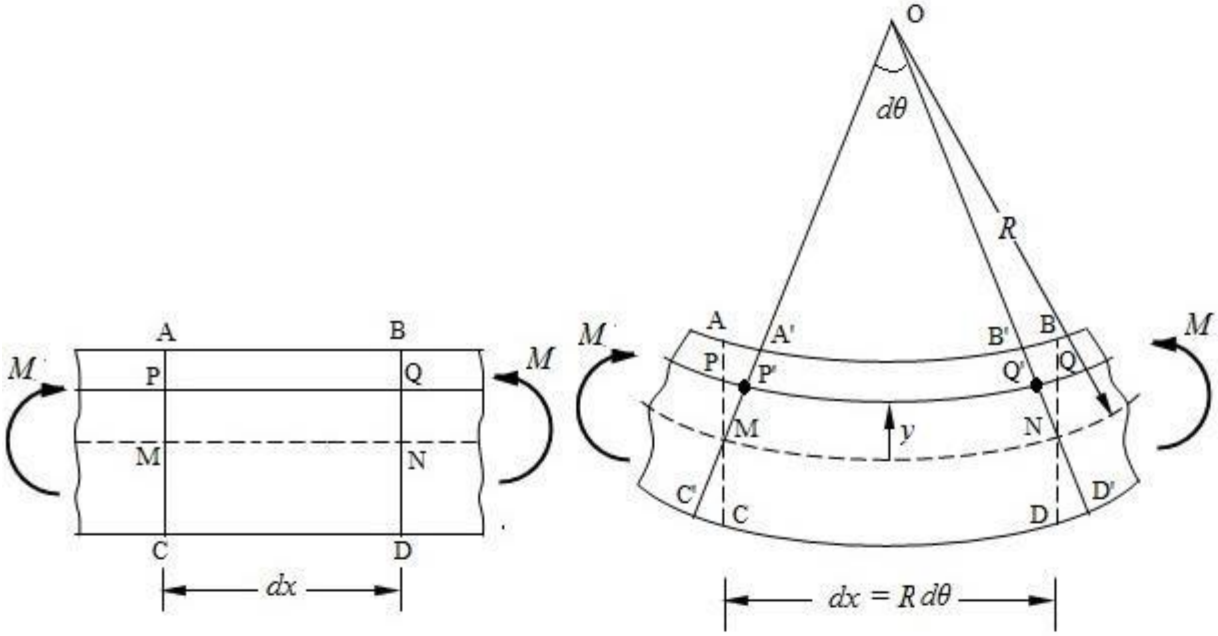
Consider a part of beam *ABCD* of length *dx* subjected to pure bending of bending moment *M* as shown in the Fig. 11. As the beam is subjected to pure bending, it bends into a circular arc.

The topmost layer *AB* is contracted to *A'B'*. The layer *PQ* below it is compressed to a lesser degree than it. The bottom most layer *CD* is elongated to *C'D'*. All other layers are subjected to different degrees of elongation or contraction degrees depending upon their position. However, one layer *MN* has not suffered any change in its length. This layer is called the ***neutral layer*** or ***neutral surface***.

Let $d\theta$ be the angle formed by the planes *A'C'* and *B'D'* and *R* be the radius of the neutral layer. Consider a fibre *PQ* at a distance of y from the neutral layer.

Original length of the fibre, $PQ = dx = Rd\theta$

After deformation, the length of the fibre is compressed to P'Q'.



(a) Beam before bending

(b) Beam after bending

Fig. 11

Decrease in length of the fibre PQ $= PQ - P'Q'$

$$= Rd\theta - (R - y)d\theta$$

$$= y\,d\theta$$

Let the projection of $C'A'$ and $D'B'$ meet at O.

Strain in the fibre PQ, $\varepsilon = \dfrac{\text{Decrease in length}}{\text{Original length}}$

$$\therefore\ \varepsilon = \frac{y\,d\theta}{R\,d\theta} = \frac{y}{R}$$

Let σ be the stress in the fibre PQ.

Then, $\varepsilon = \dfrac{\sigma}{E}$, where E is the Modulus of elasticity of the material.

$$\therefore\ \varepsilon = \frac{\sigma}{E} = \frac{y}{R}$$

$$\therefore\ \sigma = \frac{E}{R} \times y$$

Hence, the stress intensity in any fibre is proportional to the distance of the fibre from the neutral layer.

Position of Neutral Axis

Consider a beam of arbitrary cross-section as shown in the Fig. 12. Consider an elemental are δa at a distance y from the neutral axis. Let the bending stress on the element be σ.

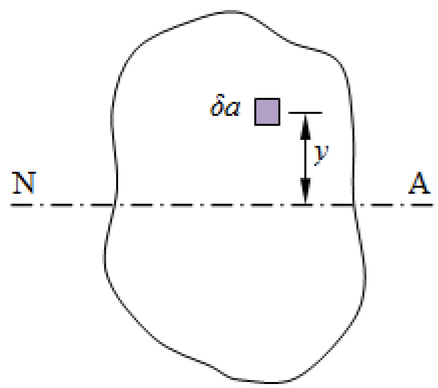


Fig. 12

Force on the elemental area $= \sigma\ \delta a$

Force over the entire cross-section of the beam $= \sum \sigma\ \delta a$

We also know, $\sigma = \frac{E}{R} \times y$

Substituting the value of σ, we get

Force over the entire cross-section of the beam $= \sum \frac{E}{R} y\delta a = \frac{E}{R} \sum y\delta a$

Since there is no axial force on the beam, from equilibrium consideration, the above axial force should be zero.

Hence, $$\frac{E}{R} \sum y\delta a = 0$$

Since, $\frac{E}{R}$ is constant for a given section, we have $\sum y\delta a = 0$

We know, $$A\,\bar{y} = \sum y\delta a$$

Where, A is the area of cross-section of the beam.

So, $$A\,\bar{y} = 0$$

$$\text{or} \qquad \bar{y} = 0$$

$\bar{y}$ is the distance of the centroid from the neutral axis. Hence, the neutral axis of the section coincides with the centroid of the section. Thus, to locate the neutral axis of a section, the centroid of the section should be determined. The line passing through the centroid, parallel to the plane of bending is the neutral axis of the beam section.

Relationship between Moment and Radius of Curvature

Consider an elemental area δa from the neutral axis of a beam section as shown in the Fig. 13.

The stress on the elemental area, $\sigma = \frac{E}{R} y$

Force on the elemental are $\sigma\, \delta a = \frac{E}{R} y\, \delta a$

Moment of resistance offered by this elemental area about the neutral axis

$$= \left(\frac{E}{R} y\, \delta a \right) y = \frac{E}{R} y^2 \delta a$$

Total moment of resistance, M offered by the cross-sectional area of beam,

$$M = \sum \frac{E}{R} y^2 \delta a$$

$$M = \frac{E}{R} \sum y^2 \delta a$$

But, $\sum y^2 \delta a$ is the moment of inertia I of the beam section about the neutral axis.

$$\therefore \quad M = \frac{E}{R} I$$

$$\frac{M}{I} = \frac{E}{R}$$

We have earlier seen that, $\frac{\sigma}{y} = \frac{E}{R}$

Combining the two equations, we get

$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$, which is known as the *bending equation*.

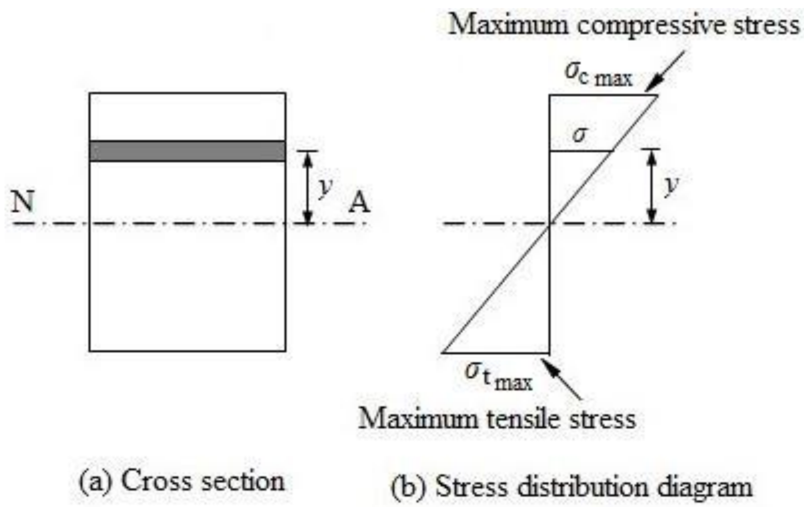


Fig. 13

Where, M = bending moment at a section,

I = moment of inertia of the beam section,

σ = stress at any layer of the beam,

y = distance of the layer from the neutral axis,

E = Young's Modulus and

R = radius of curvature.

M and I are constants for a particular beam section. Hence, σ varies proportionally to the distance y. So, maximum stress occurs at extreme fibres. The stress distribution will be triangular as shown in the Fig. 13.

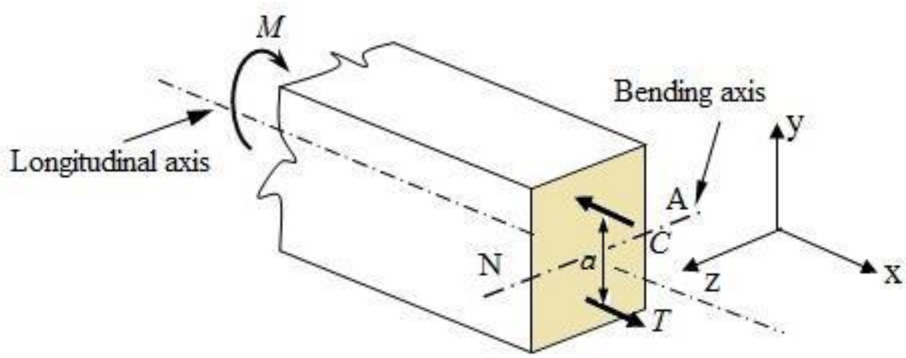


Fig. 14

The formula for flexural stress derived as above applies only to cases where the material behaves elastically. The important concepts used in deriving the flexural formula may be summed up as follows.

1. Strains in different layers of beam vary linearly with their distances from the neutral axis.
2. Properties of materials are used to relate strain and stress.
3. Equilibrium conditions are used to locate the neutral axis and to determine the internal stresses.

The internal bending moment developed by the induced flexural stresses due to bending at a section is known as ***moment of resistance*** of the section. For equilibrium of the section, the moment of resistance of a section should be equal to or greater than the applied external moment.

Flexural rigidity:

From equation of flexure, we have

$$\frac{M}{I} = \frac{E}{R}$$

$$EI = MR$$

EI is known as flexural rigidity. ***Flexural rigidity*** is the measure of flexural strength of a beam section. Higher is the flexural rigidity better is the flexural strength. It depends upon the material as well as the geometric property of the section. Elastic modulus, E reflects the material character and moment of inertia, I reflects the geometric characteristic

Economical section

In a beam of rectangular or circular section, the fibres near neutral axis are under-stressed compared to those at the top and bottom. As a result, a large portion of the beam cross-section remains under-stressed and under utilized for resisting flexure or bending.

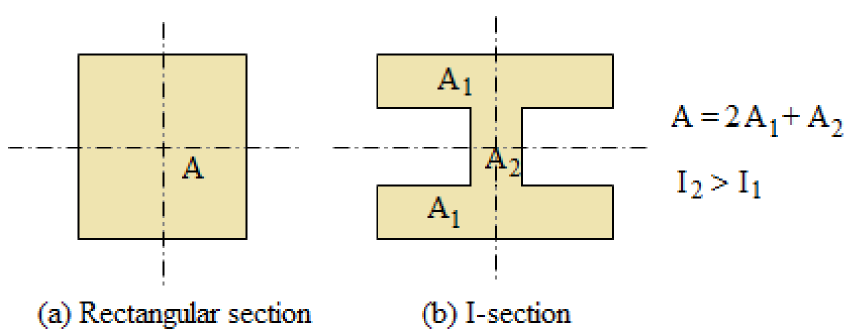


Fig. 15

The expression $M = \frac{\sigma}{y} I$ indicates that moment of resistance of a section can be greatly increased by increasing the moment of inertia by rearranging or redistributing the area while keeping the cross-sectional area and the depth of the beam unchanged. This can be achieved by changing the geometry of the section so as to spread the area farther from the neutral axis.

In order to increase the moment of resistance to bending of a beam section, it is advisable to use sections which have large area away from the neutral axis. Hence, I-section and T-sections are preferable to rectangular section.

Sections of different geometry, (i) rectangular section and (ii) I-section of equal cross-sectional area and same depth are shown in the Fig. 15.

Moment carrying capacity of a section:

From equation of flexure, we have

$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M}{I} y$$

It is obvious that bending stress is maximum on the extreme fibre at the top and bottom of the beam where y is maximum. In design of beam, the extreme fibre stress should not be allowed to exceed the allowable or permissible stress of the material. If σ_{allow} is the allowable stress for bending, then for safe design

$$\sigma_{\max} \leq \sigma_{allow}$$

$$\frac{M}{I} y_{\max} \leq \sigma_{allow}$$

If M is taken as the maximum moment carrying capacity of the section,

$$\frac{M}{I} y_{\max} \leq \sigma_{allow}$$

$$M \le \frac{I}{y_{\max}} \sigma_{allow}$$

The moment of inertia I and the extreme fibre distance $y_{\max}$ are the geometrical properties of the section. The ration of the moment of inertia and the extreme fibre distance $(I/y_{\max})$ for a given cross-section of beam is constant and is known as ***section modulus*** (Z). Thus the moment carrying capacity of a beam is given by

$$M = \sigma_{allow} Z$$

If σ_{allow} in tension and compression are same, doubly symmetric section is selected. Doubly symmetric section means a section which is symmetric about the vertical as well as neutral axis. If σ_{allow} in tension and compression are different, un-symmetric cross-section is selected such that the distance to the extreme fibers are nearly the same ratio as the respective allowable stresses. In the latter case, the moment carrying capacity in tension and compression are found separately and the smaller one is taken as the moment carrying capacity of the section.

Section Modulus of Sections of Standard Geometry

1. Rectangular section

Let us consider a rectangular section of width b and depth d as shown in the Fig. The neutral axis coincides with the centroidal axis of the beam.

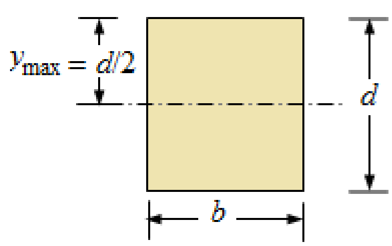


Fig. 16

Moment of inertia about the neutral axis, $I = \frac{bd^3}{12}$

Distance of outermost fibre from the neutral axis, $y_{\max} = \frac{d}{2}$

Section modulus, $$Z = \frac{I}{y_{\max}} = \frac{bd^3}{12} \times \frac{2}{d}$$

$$= \frac{1}{6} bd^2$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \frac{1}{6}\sigma bd^2$

2. Hollow Rectangular section

Let us consider a hollow rectangular section of size $B \times D$ with a symmetrical opening $b \times d$ as shown in the Fig. 17.

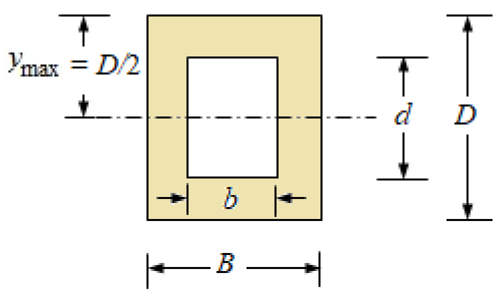


Fig. 17

Moment of inertia about the neutral axis, $I = \frac{BD^3}{12} - \frac{bd^3}{12}$

Distance of outermost fibre from the neutral axis, $y_{max} = \frac{D}{2}$

Section modulus, $Z = \frac{I}{y_{max}} = \frac{BD^3 - bd^3}{12} \times \frac{2}{D}$

$$= \frac{1}{6}\frac{\left(BD^3 - bd^3\right)}{D}$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \frac{1}{6}\sigma \frac{(BD^3 - bd^3)}{D}$

3. Circular section

Let us consider a circular section of diameter *d* as shown in the Fig. 18.

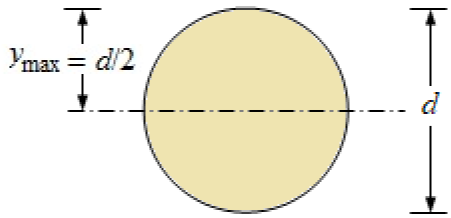


Fig. 18

Moment of inertia about the neutral axis, $I = \frac{\pi d^4}{64}$

Distance of outermost fibre from the neutral axis, $y_{max} = \frac{d}{2}$

Section modulus, $$Z = \frac{I}{y_{max}} = \frac{\pi d^4}{64} \times \frac{2}{d}$$

$$= \frac{\pi d^3}{32}$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \sigma \frac{\pi d^3}{32}$

4. Hollow Circular section

Let us consider a hollow circular section of external and internal diameter D and d respectively as shown in the Fig. 19.

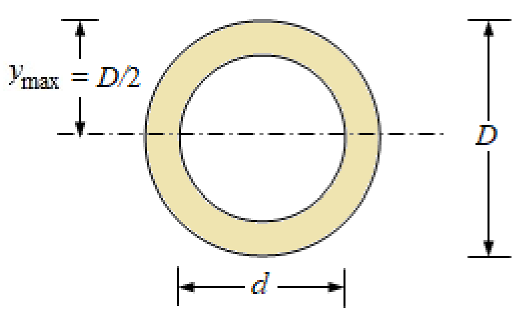


Fig. 19

Moment of inertia about the neutral axis, $I = \frac{\pi}{64}\left(D^4 - d^4\right)$

Distance of outermost fibre from the neutral axis, $y_{max} = \frac{D}{2}$

Section modulus,

$$Z = \frac{I}{y_{max}} = \frac{\pi}{64}\left(D^4 - d^4\right) \times \frac{2}{D}$$

$$= \frac{\pi}{32D}\left(D^4 - d^4\right)$$

Let σ is the maximum bending stress developed at the outermost layer.

Moment of resistance, $M = \sigma Z = \frac{\sigma\pi}{32D}\left(D^4 - d^4\right)$

Design of beam for bending

Design of beam involves the determination of the size (cross-section) of the beam for given loading condition. The maximum bending moment of the beam is determined from the loading condition. Given the bending moment and permissible bending stress of the material of the beam, the section modulus of the beam is determined from the expression of bending stress. Once the section modulus is known, width and depth can easily determined assuming the depth to width ratio.

Beam of uniform strength

In practice, a beam of uniform cross section is designed for moment of resistance same as the maximum bending moment that the beam is supposed to carry. Hence, the material in all sections except the section of maximum bending moment remains under-stressed and underutilized. Although practical, such a beam is uneconomical. Ideally, a beam of varying cross-section should be designed so that all sections attain the maximum permissible stress simultaneously. A beam in which permissible stress at all sections is reached simultaneously under a given loading, is called a ***beam of uniform strength***.

A beam of uniform strength can be obtained in different ways

a) By varying the width of beam and keeping the depth constant
b) By varying the depth of beam and keeping the width constant
c) By varying both width and depth

By varying the width of beam and keeping the depth constant

Derive the formula for cross section of a rectangular beam of uniform strength for a cantilever beam of length L carrying concentrated load at free end by keeping the depth constant.

Consider a cantilever beam of length L and uniform depth d carrying a concentrated load W at its free end as shown in the Fig. 20. Let the width varies from a minimum at its free end to a maximum of b near the fixed end.

It is obvious that the bending moment varies from minimum zero at the free end to maximum at WL at the fixed support.

Bending moment at any section at a distance of x from the free end,

$$M = Wx$$

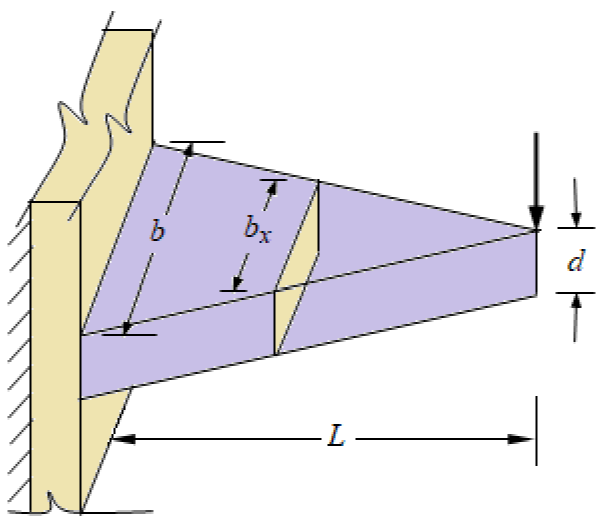


Fig. 20

From expression of flexure, we xhave

$$M = \sigma Z$$

$$Wx = \sigma Z$$

Where σ is the maximum stress at every section of the beam.

If b_x width at any section XX, then $Z = \frac{b_x d^2}{6}$

$$\therefore \qquad \sigma = \frac{Wx}{\frac{b_x d^2}{6}} = \frac{6Wx}{b_x d^2}$$

Similarly, maximum stress at support, $\sigma = \dfrac{6WL}{bd^2}$

Equating equation () and (), we have

$$\frac{6Wx}{b_x d^2} = \frac{6WL}{bd^2}$$

$$b_x = b\left(\frac{x}{L}\right)$$

At free end, i.e., $x = 0$, the width of beam $b_0 = 0$

At the fixed end, i.e., $x = L$, the width $b_L = b\left(\frac{L}{L}\right) = b$

By varying the depth of beam and keeping the width constant

Consider a cantilever beam of length L and uniform width b carrying a concentrated load W at its free end as shown in the Fig. 20. Let the depth varies from a minimum at its free end to a maximum of d near the fixed end.

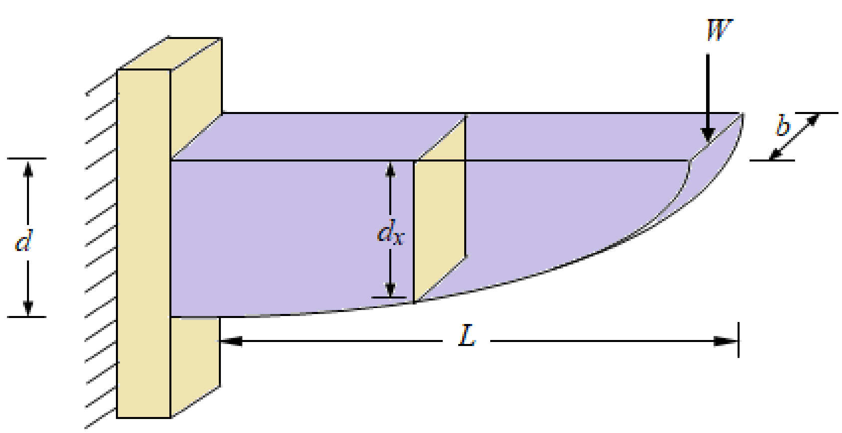


Fig. 21

It is obvious that the bending moment varies from minimum zero at the free end to maximum at WL at the fixed support.

Bending moment at any section at a distance of x from the free end,

$$M = Wx$$

From expression of flexure, we xhave

$$M = \sigma Z$$

$$Wx = \sigma Z$$

Where σ is the maximum stress at every section of the beam.

If b_x width at any section XX, then $Z = \frac{bd_x^2}{6}$

$$\therefore \qquad \sigma = \frac{Wx}{\frac{bd_x^2}{6}} = \frac{6Wx}{bd_x^2}$$

Similarly, maximum stress at support, $\sigma = \frac{6WL}{bd^2}$

Equating equation () and (), we have

$$\frac{6Wx}{bd_x^2} = \frac{6WL}{bd^2}$$

$$d_x = d\sqrt{\left(\frac{x}{L}\right)}$$

At free end, i.e., $x = 0$, the depth of beam, $d_0 = 0$

At the fixed end, i.e., $x = L$, the depth, $d_L = d\sqrt{\left(\frac{L}{L}\right)} = d$

Numerical

1. A rectangular beam of breadth 100 *mm* and depth 200 *mm* is simply supported over a span of 4 *m*. The beam is loaded with an uniformly distributed load of 5 *kN/m* over the entire span. Find the maximum bending stresses.

Solution:

Breadth of the beam, $b = 100\ mm$

Depth of beam, $d = 200\ mm$

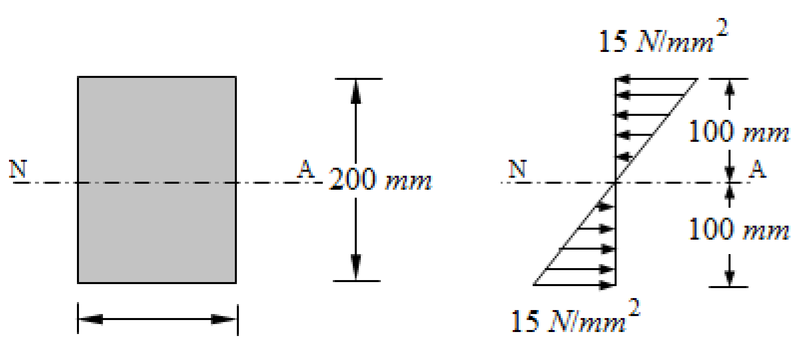
Moment of inertia, $I = \frac{1}{12}bd^3 = \frac{1}{12}\times 100\times (200)^3 = 66.67\times 10^6\ mm^4$

Span of beam, $l = 4\ m$

Uniformly distributed load, $w = 5\ kN/m$

Maximum bending moment at centre of beam, $M = \frac{wl^2}{8} = \frac{5\times 4^2}{8}$

$$= 10\, kN.M = 10^7\ N.mm$$



(a) Cross section of beam (b) Bending stress distribution

Fig. 22

Neutral axis passes through the centroid of section.

The distance of top and bottom fibre from the neutral axis, $y = 100\ mm$

Thus, maximum bending stress, $\sigma = \frac{M}{I}y = \frac{10^7}{66.67\times 10^6}\times 100$

$$= 15\, N/mm^2$$

2. A beam of I-section shown in Fig. 23 is simply supported over a span of 10 *m*. It carries a uniform load of 4 *kN*/*m* over the entire span. Evaluate the maximum bending stresses.

Solution:

Moment of inertia, $I = \frac{1}{12}\left(BD^3 - bd^3\right) = \frac{1}{12}\left(300 \times 660^3 - 280 \times 600^3\right)$

$$= 21.474 \times 10^8\ mm^4$$

Span of the beam, $l = 10\ m$

Uniformly distributed load, $w = 4\ kN/m$

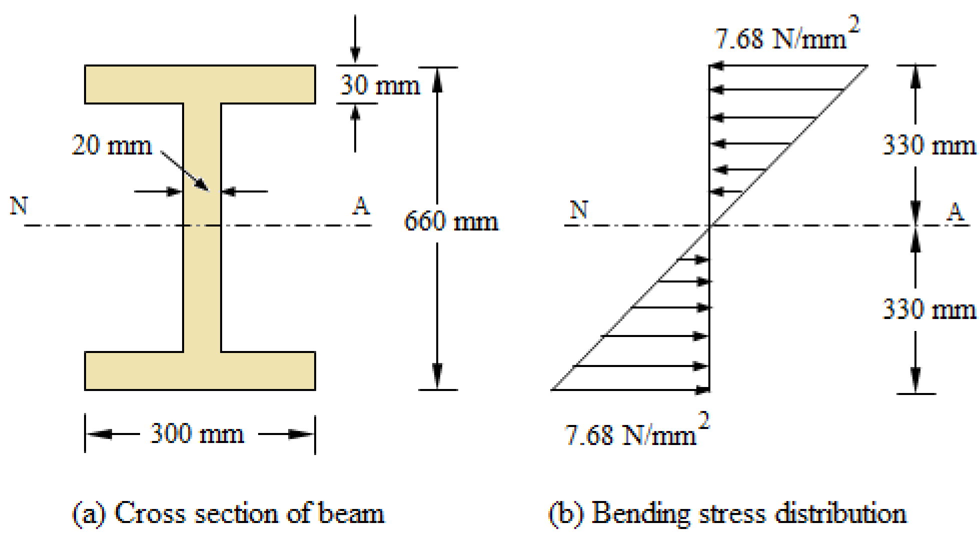


Fig. 23

Maximum bending moment at centre of beam, $M = \frac{4 \times 10^2}{8} = 50\, kN.m$

$$= 5 \times 10^7\ N.mm$$

Neutral axis passes through the centroid of I-section.

The distance of top and bottom fibre from the neutral axis, $y = 330\ mm$

Thus, maximum bending stress, $\sigma = \frac{M}{I} y = \frac{5 \times 10^7}{21.474 \times 10^8} \times 330 = 7.68 \, N/mm^2$

The bending stress at top and bottom fibres $= 7.68 \times 10^8 \, N/mm^2$

3. A beam of an I-section shown in Fig. 24 is simply supported over a span of 4 *m*. Find the uniformly distributed load the beam can carry if the bending stress is not to exceed 100 N/mm^2.

Solution:

Moment of inertia, $I = \frac{1}{12}\left(BD^3 - bd^3\right) = \frac{1}{12}\left(200 \times 300^3 - 180 \times 260^3\right)$

$$= 180.36 \times 10^6 \, mm^4$$

Maximum bending stress, $\sigma_{\max} = 100 \, N/mrn^2$

Span of beam, $l = 4 \, m$

Extreme fibre distance, $y_{\max} = 150 \, mm$

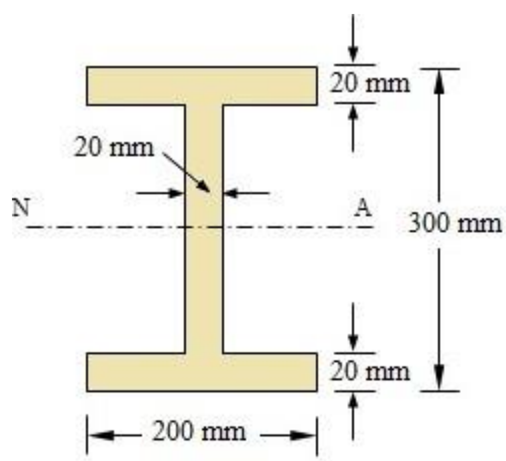


Fig. 24

Section modulus, $Z = \frac{I}{y_{\max}} = \frac{180.36 \times 10^6}{150} = 1242400 \, mm^3$

Maximum bending moment, $M = \sigma_{\max} Z = 100 \times 1242400$

$$= 124240000 \, N.mm$$

$$= 124.24\ kN.m$$

But $$M = \frac{wl^2}{8}$$

$$124.24 = \frac{w \times (4)^2}{8}$$

$$w = \frac{124.24 \times 8}{16} = 64.12\ kN/m$$

The maximum uniformly distributed load the beam can carry = 64.12 *kN/m*.

4. A timber beam of rectangular section carries a load of 2 *kN* at mid-span. The beam is simply supported over a span of 3.6 *m*. If the depth of section is to be twice the breadth, and the bending stress is not to exceed 9 N/mrn^2, determine the cross-sectional dimensions.

Solution:

Span of the beam, $l = 3.6\ m$

Uniformly distributed load, $w = 2\ kN$

Allowable bending stress, $\sigma_{\text{allow}} = 9\ N/mm^2$

Maximum bending moment at centre of beam, $M = \frac{WL}{4} = \frac{2 \times 3.6}{4} = 1.8\ kN.m$

$$= 1.8 \times 10^6\ N.mm$$

From the flexural relationship, we have $$Z = \frac{M}{\sigma_{allow}}$$

$$\frac{1}{6} bd^2 = \frac{1.8 \times 10^6}{9}$$

$$bd^2 = \frac{1.8 \times 10^6}{9} \times 6 = 1.2 \times 10^6$$

Depth of section is to be twice the breadth, i.e., $d = 2b$

So, we have $$b(2b)^2 = 1.2 \times 10^6$$

$$b^3 = \frac{1.2\times10^6}{4} = 0.3\times10^6$$

$$b = 64.94\,mm$$

$$d = 2\times 64.943 = 129.886\,mm$$

Therefore, width of beam = 65 *mm*, and depth of beam = 130 *mm*

5. A rectangular beam of width 200 *mm* and depth 300 *mm* is simply supported over a span of 5 *m*. Find the safe uniformly distributed load that the beam can carry per metre length if the allowable bending stress in the beam is 100 N/mm^2.

Solution:

Span of beam, $l = 5\ m$

Width Breadth of the beam, $b = 100\ mm$

Depth of beam, $d = 200\ mm$

Allowable bending stress, $\sigma_{allow} = 100\ N/mm^2$

Section modulus, $Z = \frac{1}{6}bd^2 = \frac{1}{2}\times200\times300^2 = 3\times10^6\ mm^3$

Moment of resistance of the beam, $M = \sigma_{allow} Z = 100\times3\times10^6$

$$= 300\times10^6\ N.mm = 300\,kN.m$$

Maximum bending moment at the centre of the beam,

$$M = \frac{wl^2}{8}$$

$$300 = \frac{w\times(5)^2}{8}$$

$$\therefore \qquad w = \frac{300\times8}{25} = 96\,kN.m$$

So, the load that the beam can carry is 96 *kN/m*.

6. A rectangular beam of size 60 *mm* x 100 *mm* has a central rectangular hole of size 15 *mm* x 20 *mm*. The beam is subjected to bending and the maximum bending stress is limited to 100 N/mm^2. Find the moment of resistance of the hollow beam section.

Solution:

External dimension of hollow rectangular beam: B = 60 mm, D = 100 mm

Size of the central hole: b = 15 mm, d = 20 mm

Moment of inertia of the hollow beam section, $I = \frac{1}{12}\left(BD^3 - bd^3\right) = \frac{1}{12}\left(60\times100^3 - 15\times20^3\right)$

$$= 4.999\times10^6\ mm^4$$

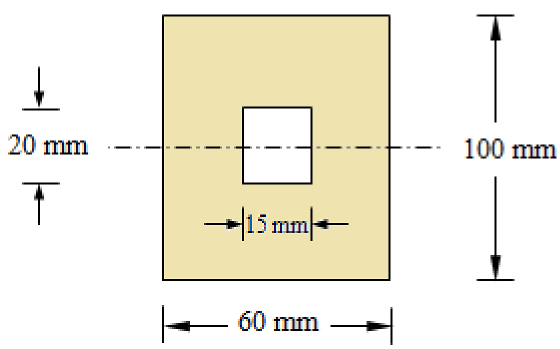


Fig.25

Extreme fibre distance, $y_{\max} = \frac{100}{2} = 50\, mm$

Section modulus, $Z = \frac{I}{y_{\max}} = \frac{4.999\times10^6}{50} = 9.98\times10^4\ mm^3$

Allowable bending stress, $\sigma_{allow} = 100\ N/mm^2$

Moment of resistance, $M = \sigma_{allow} Z = 100\times9.98\times10^4$

$$= 9.98\times10^6\ N.mm$$

$$= 9.98\, kN.mm$$

7. Find the ratio of the dimensions of the strongest rectangular beam that can be cut from a circular log of wood of diameter *D*.

Solution:

Let b be the width and d the depth of the strongest rectangular beam section as shown in the Fig. 26.

From the geometry, we have $b^2 + d^2 = D^2$

$$d^2 = D^2 - b^2$$

Section modulus of the rectangular section,

$$Z = \frac{1}{6}bd^2 = \frac{1}{6}b\left(D^2 - b^2\right)$$

$$= \frac{1}{6}\left(bD^2 - b^3\right)$$

Strongest section in bending should have largest section modulus.

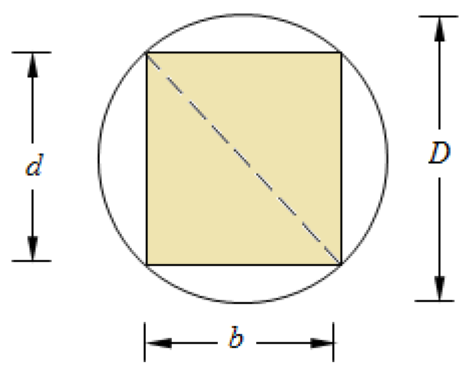


Fig. 26

Hence, $$\frac{dZ}{db} = \frac{1}{6}\left(D^2 - 3b^2\right) = 0$$

$$3b^2 = D^2$$

$$b = \frac{D}{\sqrt{3}}$$

And $$d = \sqrt{D^2 - b^2} = \sqrt{D^2 - \frac{D^2}{3}}$$

$$= \sqrt{\frac{2D^2}{3}} = \left(\sqrt{\frac{2}{3}}\right)D$$

8. Two sections of same material; one of solid circular section and the other hollow circular section of internal diameter half the external diameter, have the same flexural strength. Which one of them is economical?

Solution:

Let D = Diameter of solid circular section

D_1 = Outer diameter of hollow circular section

Inside diameter of hollow circular section, $D_2 = 0.5\ D_1$

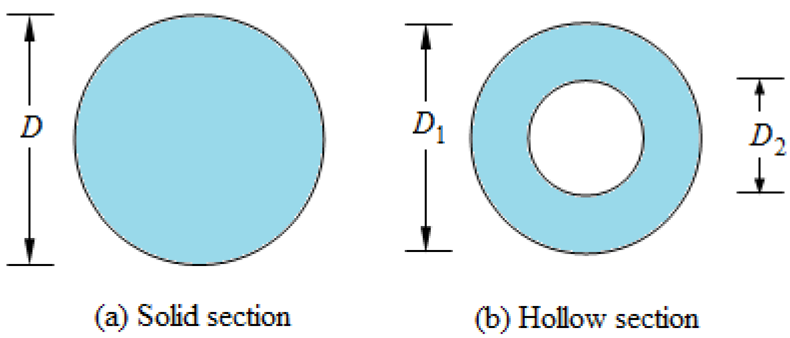


Fig. 27

Section modulus of solid section, $Z_1 = \frac{\pi}{32}D^3$

Section modulus of hollow section, $Z_2 = \frac{\pi}{32D_1}\left(D_1^4 - D_2^4\right) = \frac{\pi}{32D_1}\left\{D_1^4 - \left(0.5D_1\right)^4\right\}$

$$= \frac{\pi}{32} \times 0.9375D_1^3$$

Since both sections have same flexural strength, their section modulus should be equal.

Hence, $$\frac{\pi}{32}D^3 = \frac{\pi}{32} \times 0.9375D_1^3$$

$$D^3 = 0.9375 D_1^3$$

$$D = 0.98 D_1$$

$$\frac{\text{Cross-sectional area of solid section}}{\text{Cross-sectional area of hollow section}} = \frac{A_s}{A_h} = \frac{\frac{\pi}{4} D^2}{\frac{\pi}{4}\left(D_1^2 - D_2^2\right)} = \frac{D^2}{\left\{D_1^2 - (0.5D_1)^2\right\}}$$

$$= \frac{D^2}{0.75 D_1^2} = \frac{1}{0.75} \times \left(\frac{D}{D_1}\right)^2$$

$$= \frac{1}{0.75} \times (0.98)^2 = 1.28$$

Since the sectional area of hollow section is less than that of solid section, for a given length of the beam, the weight of hollow section will be less. Hence hollow section is economical.

9. A cantilever of 2 *m* length and square section 200 *mm* x 200 *mm*, just fails in bending when a point load of 12 *kN* is placed at its free end. A beam of rectangular cross section of same material, 150 *mm* wide and 300 *mm* deep, is simply supported over a span of 3 *m*. Calculate the maximum concentrated load that the beam can carry at its centre without failure.

Solution:

The two beams with loading conditions are shown in the Fig.

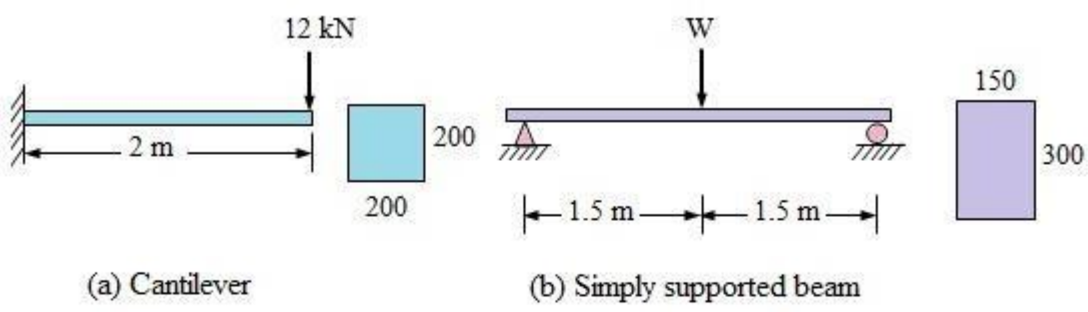


Fig. 28

Maximum bending moment in cantilever beam, $M_c = 12 \times 2 = 24\, kN.m$

$$= 24 \times 10^6\ N.mm$$

Let σ_{allow} is the stress at which the beam fails, $M_c = \sigma_{allow} Z = \frac{1}{6} bd^2 \sigma_{allow}$

$$\frac{1}{6} \times 200 \times 200^2 \times \sigma_{allow} = 24 \times 10^6$$

$$\sigma_{allow} = 18\, N/mm^2$$

Let $W\, kN$ be the maximum central concentrated that the beam can carry without failure.

Maximum bending moment at the mid span, $M_s = \frac{WL}{4} = \frac{W \times 3}{4} = 0.75W\, kN.m$

$$= 0.75 \times 10^6\, W\, N.mm$$

Moment of resistance of simply supported beam section,

$$M_R = \sigma_{allow} Z = 18 \times \frac{1}{6} \times 150 \times 300^2$$

$$= 40.5 \times 10^6\ N.mm$$

Equating maximum bending moment (M_s) to moment of resistance (M_R), we have

$$0.75 \times 10^6\, W = 40.5 \times 10^6$$

$$W = 54\, kN$$

10. For a given sectional area, compare the moments of resistance of circular and square section.

Solution:

Let the diameter of the circular section be d.

Area of circular section, $A = \frac{\pi}{4} d^2$

Section modulus, $Z_C = \frac{\pi}{32} d3$

Let the square section has side of a.

Since both circular and square section have the same area,

$$a^2 = \frac{\pi}{4} d^2$$

$$a = \frac{\sqrt{\pi}}{2} d$$

Section modulus of square section, $Z_S = \frac{a^3}{6} = \frac{\pi\sqrt{\pi}}{48} d^3$

Ratio of Section modulus of square section and circular section,

$$\frac{Z_S}{Z_C} = \frac{\frac{\pi\sqrt{\pi}}{48} d^3}{\frac{\pi}{32} d^3} = 1.18$$

Hence, flexural strength of square section is 1.18 times more than that of circular section of equal area.

11. Compare the moments of resistance of a square section of given material when the beam section is placed such that (i) two sides are parallel and (ii) one diagonal vertical.

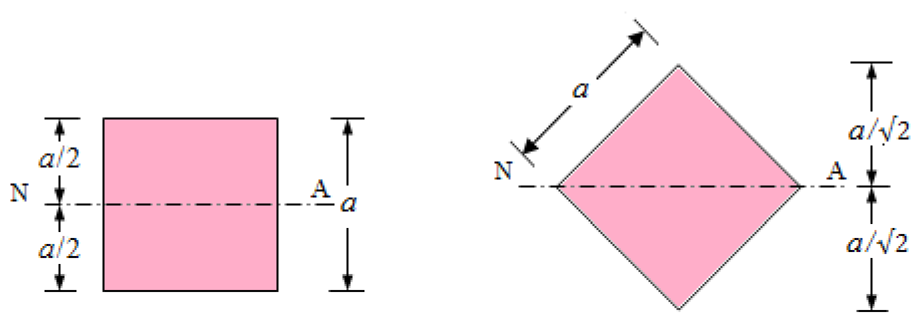
Solution:

Square section with two sides horizontal is shown in the Fig. 29(a).

Section modulus of square section with two sides horizontal, $Z_1 = \frac{a^3}{6}$

Let σ is the permissible flexural stress.

Moment of resistance, $M_1 = \sigma Z_1 = \frac{\sigma a^3}{6}$



(a) Sides parallel (b) Diagonal vertical

Fig. 29

Square section with on diagonal vertical is shown in the Fig. 29(b).

Moment of inertia about the neutral axis, i.e., the diagonal of the square section = Twice the moment of inertia of triangle of base $\sqrt{2}a$ and height $a/\sqrt{2}$.

$$I_2 = 2 \times \frac{\sqrt{2}a\left(\frac{a}{\sqrt{2}}\right)^3}{12} = \frac{a^4}{12}$$

Extreme fibre distance, $$y_{\max} = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}$$

Section modulus of square section with one diagonal vertical,

$$Z_2 = \frac{I_2}{y_{\max}} = \frac{\frac{a^4}{12}}{\frac{a}{\sqrt{2}}} = \frac{\sqrt{2}a^3}{12}$$

Moment of resistance, $$M_2 = \sigma Z_2 = \frac{\sqrt{2}\sigma a^3}{12}$$

Ration of the moments of resistance of section in two different positions,

$$\frac{M_1}{M_2} = \frac{\frac{\sigma a^3}{6}}{\frac{\sqrt{2}\sigma a^3}{12}} = \sqrt{2} = 1.414$$

12. Three beams of same material with circular, square and rectangular cross sections have the same length and are subjected to same maximum bending moment. The depth of the rectangular section is twice the width. Compare their weights.

Solution:

Fig. 30 shows three different sections, circular, square, and rectangular of beam.

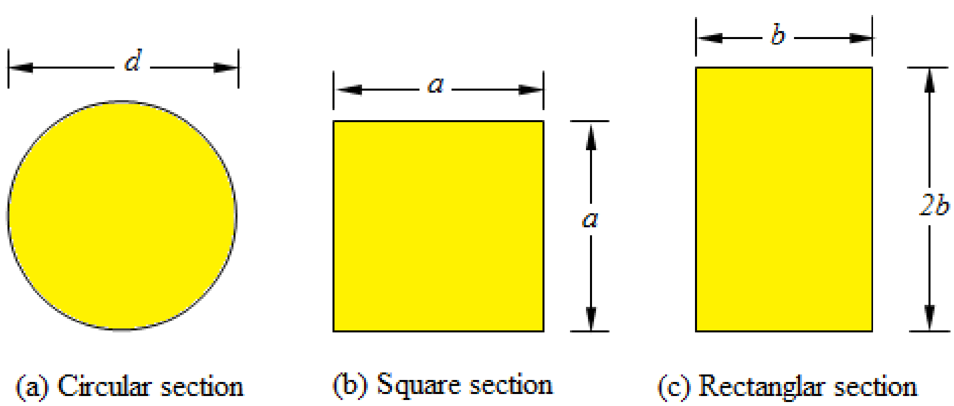


Fig. 30

Let Diameter of circular section = d,

Side of square section = a, and

Width and depth of rectangular section are b and $2b$ respectively.

As beams of three different cross sections of equal allowable stress are subjected same maximum bending moment, they must have same strength. Hence, all sections should have equal section modulii.

Section modulus of circular section, $Z_C = \dfrac{\pi d^3}{32}$

Section modulus of circular section, $Z_S = \dfrac{a^3}{6}$

Section modulus of circular section, $Z_R = \dfrac{b(2b)^2}{6} = \dfrac{2}{3}b^3$

We have $$\frac{\pi d^3}{32} = \frac{a^3}{6} = \frac{2}{3}b^3$$

$\therefore \quad d = 1.193a$ and $b = 0.6299a$

$$\frac{\text{Weight of circular beam}}{\text{Weight of square beam}} = \frac{\text{Area of circular section}}{\text{Area of square section}} = \frac{\frac{\pi d^2}{4}}{a^2} = \frac{\pi}{4}\left(\frac{d}{a}\right)^2$$

$$= \frac{\pi}{4}(1.193)^3 = 1.118$$

$$\frac{\text{Weight of rectangula r beam}}{\text{Weight of square beam}} = \frac{\text{Area of rectangula r section}}{\text{Area of square section}} = \frac{2b^2}{a^2} = 2\left(\frac{b}{a}\right)^2$$

$$= 2(0.6299)^2 = 0.7936$$

13. A beam of symmetric *I*-section has flange size 100 mm x 15 mm, overall depth 250 mm. Thickness of web is 8 mm. Compare the flexural strength of this section with that of a beam of rectangular section of same material and area. The width of rectangular section is two-third of its depth.

Solution:

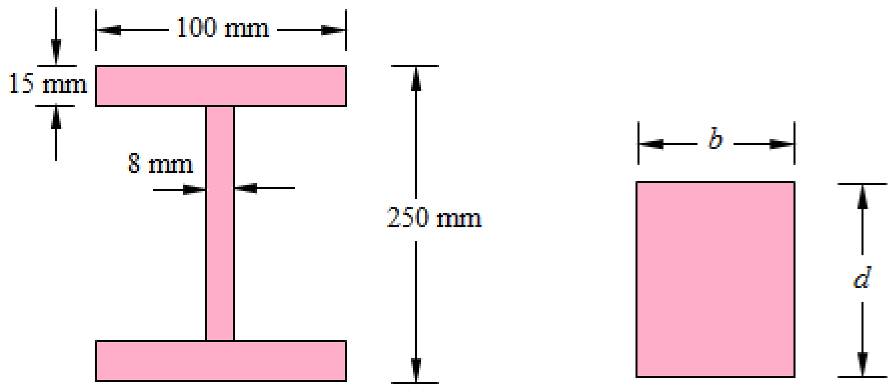
The *I*-section and the rectangular section of equal area are shown in the Fig. 31.

Area of *I*-section, $A_I = (2 \times 100 \times 15) + (220 \times 8) = 4760\,mm^2$

Moment of inertia of *I*-section, $I_I = \dfrac{100 \times 250^3}{12} - \dfrac{92 \times 220^3}{12} = 4.8574\,mm^4$

Section modulus of *I*-section, $Z_I = \dfrac{I}{y_{\max}} = \dfrac{4.8574 \times 10^7}{125}$

$= 388592\,mm^3$



(a) I-section

(b) Rctangular section

Fig. 31

Let the depth of the rectangular section = *d mm*

Width of the rectangular section, $b = \dfrac{2}{3}d$

Area of the rectangular section, $A_R = \dfrac{2}{3}d \times d = \dfrac{2}{3}d^2$

Since the area of two sections are equal, $\dfrac{2}{3}d^2 = 4760$

$d = 84.50\,mm$

and $b = \dfrac{2}{3} \times 84.50 = 56.33\,mm$

Section modulus of rectangular section, $Z_R = \dfrac{bd^2}{6} = \dfrac{56.33 \times (84.50)^2}{6}$

$= 67035mm^3$

$$\frac{\text{Flexural strength of I-section}}{\text{Flexural strength of rectangular section}} = \frac{Z_I}{Z_R} = \frac{388592}{67035} = 5.80$$

14. A cast iron beam of an *I*-section with top flange 80 *mm* x 40 *mm*, bottom flange 160 *mm* x 40 *mm* and web 120 *mm* x 20 *mm*. If the tensile stress is not to exceed 30 N/mm^2 and compressive stress 90 N/mm^2, what is the maximum uniformly distributed load the beam can carry over a simply supported span of 6 *m*, if the bottom flange is in tension?

Solution:

The cross section of the beam is as shown in the Fig. 32.

Let $\bar{y}$ is the distance of the centroid (neutral axis) from the bottom fibre (tension fibre).

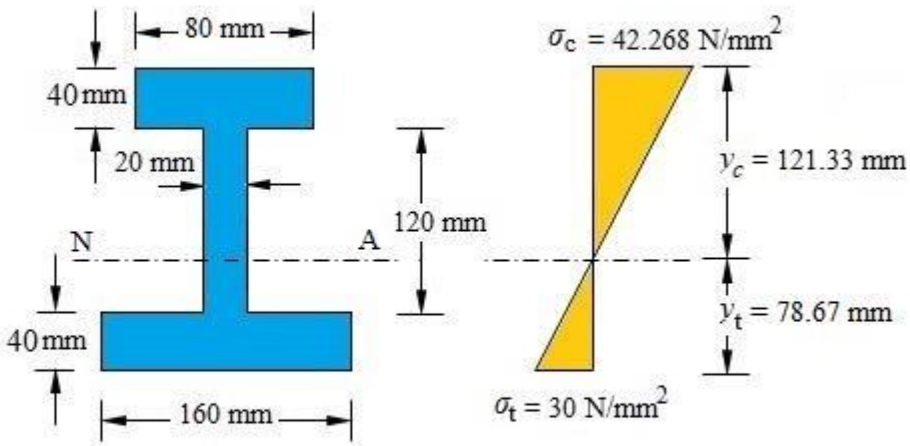


Fig. 32

$$\bar{y} = \frac{\sum a_i y_i}{A} = \frac{160\times 40\times 20 + 20\times 120\times 100 + 80\times 40\times 180}{80\times 40 + 20\times 120 + 80\times 40\times 120}$$

$$= \frac{944000}{12000} = 78.67\, mm$$

Moment of inertia,

$$I = \frac{1}{12}\times 160\times 40^3 + 160\times 40\times (78.67-20)^2 + \frac{1}{12}\times 20\times 120^3 + 20\times 120\times (100-78.67)^2$$
$$+\frac{1}{12}\times 80\times 40^3 + 80\times 40\times (180-78.67)^2$$

$$= 60138670\, mm^4$$

Tension occurs at the bottom and compression at the top.

Bottom extreme fibre distance (large flange, tension flange), $y_t = 78.67\,mm$

Top extreme fibre distance (compression flange), $y_c = 200 - 78.67 = 121.33mm$

Moment of resistance from tensile strength consideration,

$$= \sigma_{allow} \frac{I}{y_t} = 30 \times \frac{60138670}{78.67} = 22933266.81 N.mm$$

$$= 22.933\,kN.m$$

Moment of resistance from compressive strength consideration,

$$= \sigma_{allow} \frac{I}{y_c} = 90 \times \frac{60138670}{121.33} = 44609579.65\,N.mm$$

$$= 44.609\,kN.m$$

Hence, actual moment resistance is smaller of the above two, i.e., 22.993 *kN*

Maximum bending moment, $= \frac{wl^2}{8} = \frac{w \times 6^2}{8} = 4.5w$

Equating the maximum bending moment with the moment of resistance, we have

$$4.5w = 22.933$$

$$w = 5.096\,kN/m$$

Alternatively,

Suppose the maximum stress in compression at the top is 90 N/mm^2.

Corresponding maximum stress in tension at the bottom,

$$\sigma_t = \frac{y_t}{y_c} \times \sigma_c = \frac{78.67}{121.33} \times 90$$

$$= 58.355 > 30\,N/mm^2 \quad \text{(Not possible)}$$

But the permissible tensile stress is only 30 N/mm^2. Hence, let the maximum tensile stress be allowed to reach 30 N/mm^2.

Corresponding maximum compressive stress at the top,

$$\sigma_c = \frac{y_c}{y_t} \times \sigma_t = \frac{121.33}{78.67} \times 30$$

$$= 42.268\, N/mm^2 < 90\, N/mm^2 \quad (OK)$$

Hence, the beam will fail in tension at the bottom flange.

Moment of resistance from tensile strength consideration,

$$= \sigma_{allow} \frac{I}{y_t} = 30 \times \frac{60138670}{78.67} = 22933266.81 N.mm$$

$$= 22.933\, kN.m$$

Maximum bending moment, $= \frac{wl^2}{8} = \frac{w \times 6^2}{8} = 4.5w$

Equating the maximum bending moment with the moment of resistance, we have

$$4.5w = 22.933$$

$$w = 5.096\, kN/m$$

15. Two wooden planks 60 mm x 160 mm each are connected together to form a cross section of a beam as shown in the Fig. If a sagging bending moment of 3500 *N.m* is applied about the horizontal axis, find the stresses at the extreme fibre of the cross-section. Also calculate the total tensile force on the cross-section.

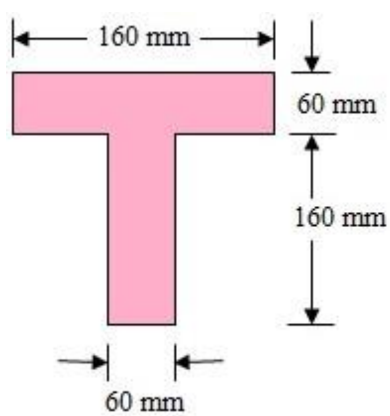


Fig.

Solution:

Let us locate the centroid and hence the neutral axis, and find moment of inertia of the section. Consider the bottom of T-section as the reference axis for location of centroid. The T-section consists of two components, web and flange.

The relevant calculations are shown in the table.

Distance of the centroidal axis GG from the bottom edge,

$$y = \frac{\sum ay}{\sum a} = \frac{2610000}{19200} = 135.94\,mm$$

Moment of inertia at the bottom edge, $I_b = \sum I_{Self} + \sum ay^2$

$$= 23.36\times10^6 + 408\times10^6 = 431.36\times10^6\ mm^4$$

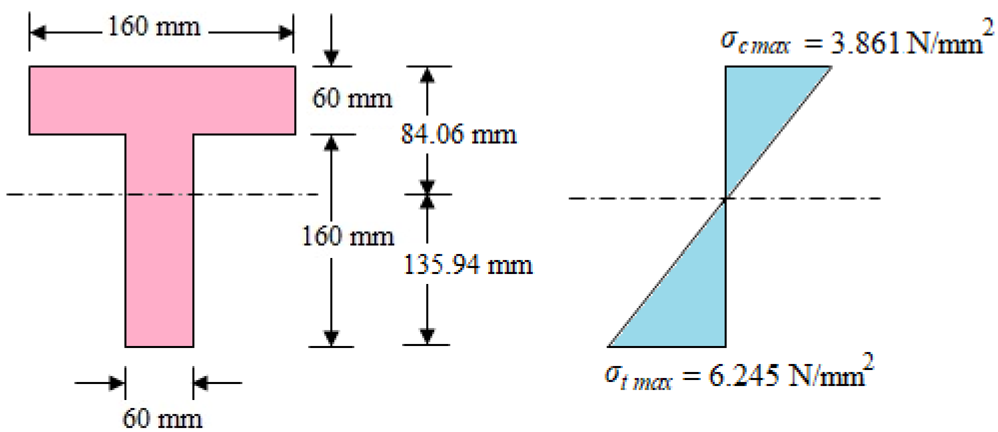


Fig.

But, $I_b = I_G + \left(\sum a\right)\bar{y}^2$

$$I_G = I_b - \left(\sum a\right)\bar{y}^2 = 431.36\times10^6 - 19200\times135.94^2$$

$$= 76190074.88\,mm^4$$

Let the maximum tensile and compressive stresses at extreme fibres be σ_{tmax} and σ_{cmax} respectively.

Components	Area a (mm^2)	Centroidal distance from the bottom edge, y (mm)	ay (mm^3)	ay^2 (mm^4)	I_{Self} (mm^4)
Web	9600	80	786000	61.44 x 10^6	$\frac{60\times160^3}{12}=20.48\times10^6$
Flange	9600	190	1824000	346.56 x 10^6	$\frac{160\times60^3}{12}=2.88\times10^6$
Total	19200		2610000	408 x 10^6	23.36 x 10^6

We have,

$$\sigma_{t\max}=\frac{M}{I}y_t=\frac{3500\times1000}{76190074.88}\times135.94$$

$$=6.245\,N/mm^2$$

$$\sigma_{c\max}=\frac{M}{I}y_c=\frac{3500\times1000}{76190074.88}\times84.06$$

$$=3.861N/mm^2$$

Total tensile force = Average tensile stress x area of tensile zone

$$=\frac{6.245}{2}\times\left(135.94\times60\right)=25468.359\,N$$

16. A water main of 1000 mm internal diameter and 10 mm thickness is running full. If the bending stress is not to exceed 56 N/mm^2, find the greatest span on which the pipe may be freely supported. Steel and water weigh 76800 N/m^3 and 10000 N/m^3 respectively.

Solution:

Internal diameter of the pipe, d = 1000 mm = 1 m

External diameter of pipe, D = 1000 + 2 x 10 = 1020 mm = 1.02 m

Consider 1 m length of the water main.

Area of the pipe section, $A=\frac{\pi}{4}\left(D^2-d^2\right)=\frac{\pi}{4}\left(1.02^2-1^2\right)$

$$=0.03173m^2$$

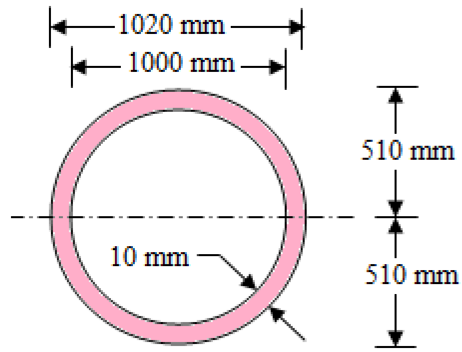


Fig.

Area of the water section, $A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 1^2$

$= 0.7854\, mm^2$

Weight of one metre length of pipe = 0.03173 x 1 x 76800 = 2493.978 N

Weight of water in one metre length of the pipe = 0.7854 x 1 x 10000 = 7854 N

Total load on the pipe per metre run = 2493.978 + 7854 = 10347.978 N

Let the maximum span of the pipe l m.

Maximum bending moment, $M = \frac{wl^2}{8} = \frac{10347.978 l^2}{8} = 1293.497 l^2 \; N.m$

$= 1293.497 \times 1000 l^2 \, N.mm$

Moment of inertia of the pipe section about the neutral axis,

$$I = \frac{\pi}{64}\left(D^4 - d^4\right) = \frac{\pi}{64}\left(1020^4 - 1000^4\right)$$

$= 4046.379 \times 10^6 \; mm^4$

We know, $\frac{M}{I} = \frac{\sigma}{y}$

$$\frac{1293.497 \times 1000 l^2}{4046.379 \times 10^6} = \frac{56}{510}$$

$$l^2 = \frac{56 \times 4046.379 \times 10^6}{510 \times 1293.497 \times 1000} = 343.494$$

$$l = 18.533m$$

Shear Force and Bending Moment

TYPES OF FORCES: Basically, structural members experience two types of forces.

External Forces: Actions of other bodies on the structure under consideration are known as external forces.

Internal Forces: Forces and couples exerted on a member or portion of the structure by the rest of the structure. Internal forces always occur in equal but opposite pairs.

TYPES OF LOAD

The following are the important types of load which act on a beam.

1. Concentrated or point load,
2. Uniformly distributed load, and
3. Uniformly varying load

1. **Concentrated or Point Load:** Load acting at a point or over very limited area compared to the length of the beam is known as concentrated load or point load.

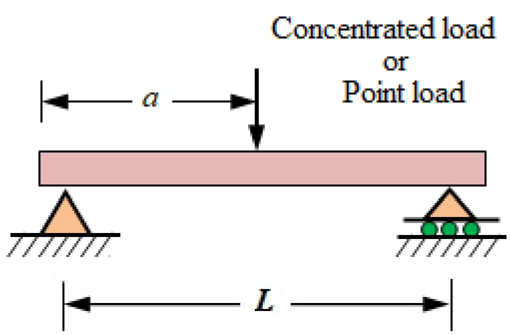


Fig. 1

2. **Uniformly Distributed Load:** Load that is spread over a beam with uniform rate of loading, ('*w*' per unit run) is known as uniformly distributed load or *UDL*. Uniformly distributed load is also known as rectangular load.

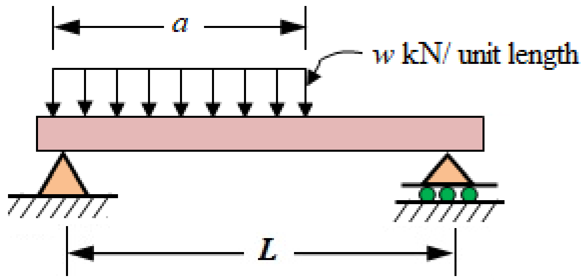


Fig. 2

3. **Uniformly Varying Load:** Load that is spread over a beam with the rate of loading uniformly from one point to the other along the beam is known as uniformly varying load. Uniformly varying distributed load is also known as triangular load.

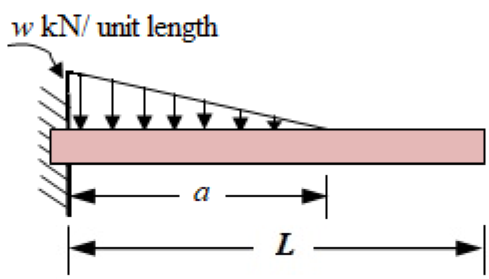


Fig. 3

4. **Parabolic Load:** If the variation of load distribution follows the equation of parabola, it is known as parabolic distributed load or simply parabolic load.

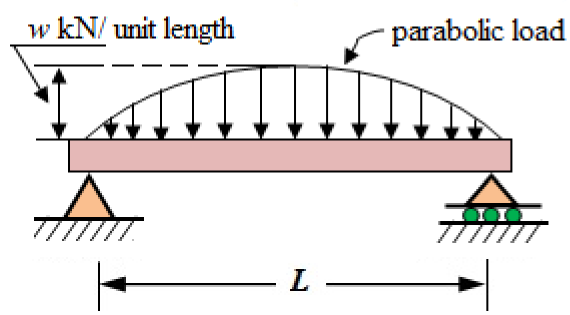


Fig. 4

TYPES OF SUPPORTS

1. Simple support
2. Roller Support

3. Pin (or) Hinge Support
4. Fixed support

Simple Supports

Simple support is just a support on which structural member rests. It is idealized to be a frictionless surface support. It only resists vertical movement of support. A simple support is free to rotate and translate along the surface upon which it rests. The resulting reaction force is always a single force perpendicular to the plane of support.

The horizontal or lateral movement allowed is up to a limited extent and after that the structure loses its support. For example, if a plank is laid across gap to provide a bridge, it is assumed that the plank will remain in its place. It will do so until a foot kicks it or moves it. At that moment the plank will move because the simple connection cannot develop any resistance to the lateral load.

This type of support is not commonly used in structural purposes. However, Simple supports are often found in zones of frequent seismic activity.

Roller Supports

Roller supports are free to rotate and translate along the surface upon which they rest. The surface can be horizontal, vertical, or sloped at any angle. They cannot resist parallel or horizontal forces and moment. They only resist perpendicular forces. Hence, the resulting reaction force is always a single force that is perpendicular to the plane of support.

This type of support is provided at one end of bridge spans. The reason for providing roller support at one end is to allow contraction or expansion of bridge deck with respect to temperature differences in atmosphere. If roller support is not provided then it will cause severe damage to the banks of bridge. But this horizontal force should be resisted by at least one support to provide stability so, roller support should be provided at one end only not at both ends.

Pinned Supports

A pinned support is same as hinged support. It can resist both vertical and horizontal forces but not a moment. It allows the structural member to rotate, but not to translate in any direction.

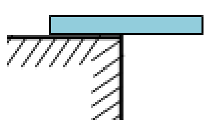
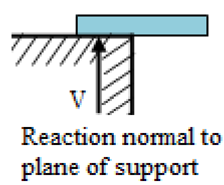
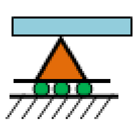
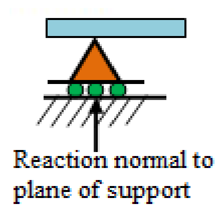
Many connections are assumed to be pinned connections even though they might resist a small amount of moment in reality. It is also true that a pinned connection could allow rotation in only one direction; providing resistance to rotation in any other direction. In human body knee is the best example of hinged support as it allows rotation in only one direction and resists lateral movements. Ideal pinned and fixed supports are rarely found in practice, but beams supported on walls or simply connected to other steel beams are regarded as pinned. The distribution of moments and shear forces is influenced by the support condition.

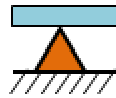
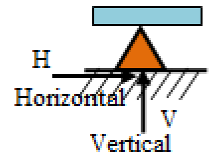
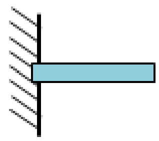
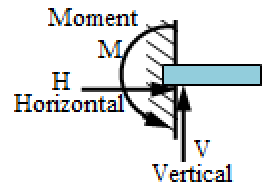
Best example for hinged support is door leaf which only rotates about its vertical axis without any horizontal or vertical movement.

Fixed Supports

Fixed support can resist vertical and horizontal forces as well as moment since they restrain both rotation and translation. They are also known as rigid support for the stability of a structure there should be one fixed support. A flagpole at concrete base is common example of fixed support In RCC structures the steel reinforcement of a beam is embedded in a column to produce a fixed support as shown in above image. Similarly all the riveted and welded joints in steel structure are the examples of fixed supports Riveted connection are not very much common now a days due to the introduction of bolted joints.

Table 1. Idealized Structural Supports

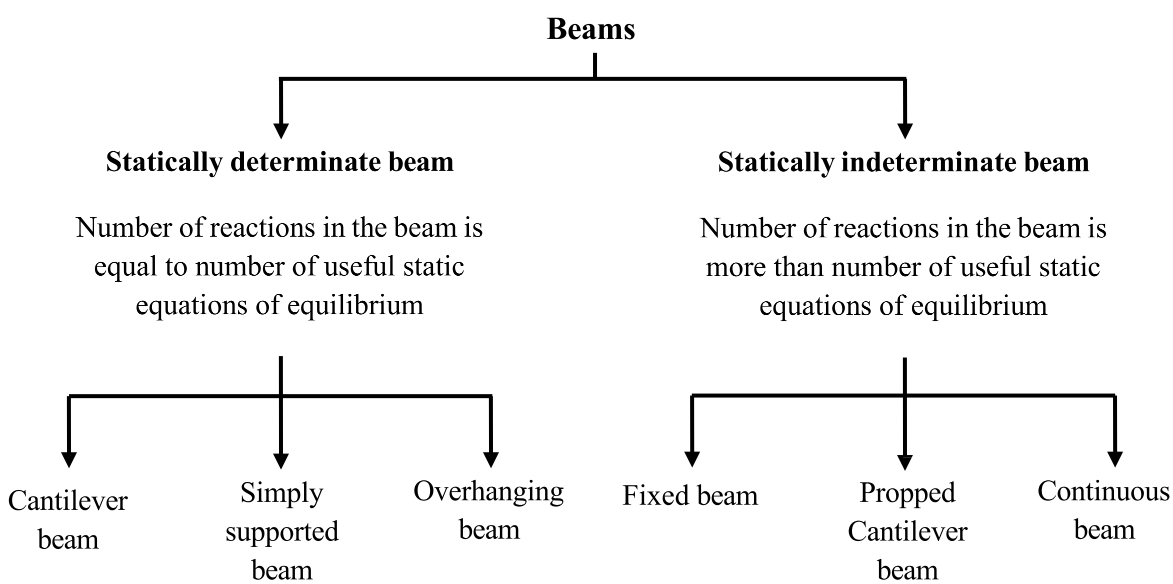
Types of supports	Real life Example	Symbol	Movement allowed and prevented	Unknown reactions
Frictionless or Simple support			Prevented: vertical translation Allowed: horizontal translation and rotation	
Roller support			Prevented: vertical translation Allowed: horizontal translation and rotation	

Pinned or hinged support			Prevented: horizontal translation and vertical translation Allowed: Rotation	
Fixed or Built-in support			Prevented: horizontal translation, vertical translation and rotation	

BEAM:

A **Beam** is defined as a structural member subjected to transverse shear loads (load normal to the axis of the beam) during its functionality. Due to the transverse shear loads, a beam is subjected to variable shear force and bending moment. Beam is a flexural member, designed primarily for bending. Analysis of beam pertains to the calculations of shear forces and bending moments along the length of the beam and drawing of shear force diagram and bending moment diagram.

TYPES OF BEAMS: Depending upon the degrees of freedom and support conditions beams are of various types.



Statically Determinate Beam

A beam is said to be statically determinate if all its reaction components can be calculated by applying three conditions of static equilibrium.

Statically Indeterminate Beam

When the number of unknown reaction components exceeds the static conditions of equilibrium, the beam is said to be statically indeterminate. To determine the unknown reactions additional equations of deformations are required.

The following are the important types of beam

1. Cantilever beam,
2. Simply supported beam,
3. Overhanging beam,
4. Fixed beams, and
5. Continuous beam.

1. Cantilever beam

A beam which is fixed or built into a rigid support at one end and free at the other end is known as cantilever beam. Such beam is shown in Fig. The built-in support prevents displacements as well as rotations of the end of the beam. Cantilever is statically determinate.

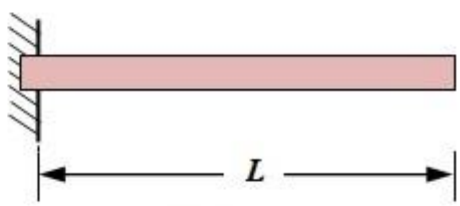


Fig.5 Cantilever beam

2. Simply Supported beam

A beam supported or resting freely on the supports at its both ends is known as simply supported beam. Such beam is shown in Fig. The end supports are free to rotate and have no moment of resistance. Simply supported beam is statically determinate beam.

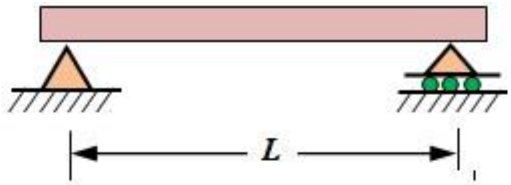


Fig. 6 Simply supported beam

3. Overhanging Beam

A beam supported over two supports and extended beyond one or both the supports is known as overhanging beam. An ***overhanging beam***, shown in Fig., is supported by a pin and a roller support, with one or both ends of the beam extending beyond the supports. It is a statically determinate beam.

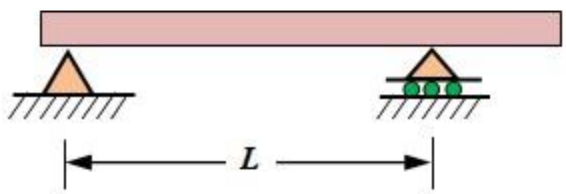


Fig. 7 Overhanging beam

4. Fixed Beam

A beam with both ends fixed or built into the supports or walls, is known as fixed beam. Such beam is shown in Fig. A fixed beam is also known as a built-in or encastred beam. It is a statically indeterminate beam.

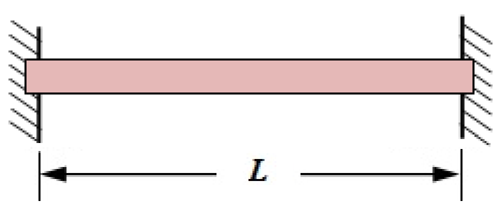


Fig. 8 Fixed beam

5. Propped cantilever beam

A beam with one end fixed and the other end simply supported over a roller is known as propped cantilever beam or simply propped cantilever. Propped cantilever is statically indeterminate.

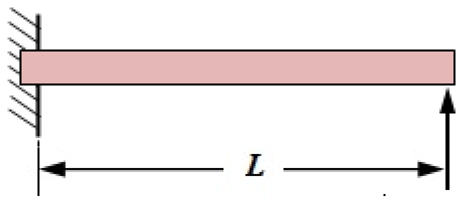


Fig.9 Propped cantilever

6. Continuous Beam

A beam which is supported over more than two supports is known as continuous beam. Continuous beam is also statically indeterminate.

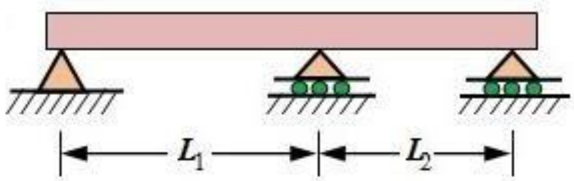


Fig.10 Continuous beam

SHEAR FORCE AND BENDING MOMENT:

The beams transfer the transverse (vertical) loads to the supports. In the process of load transfer, they experience shear force and bending moments.

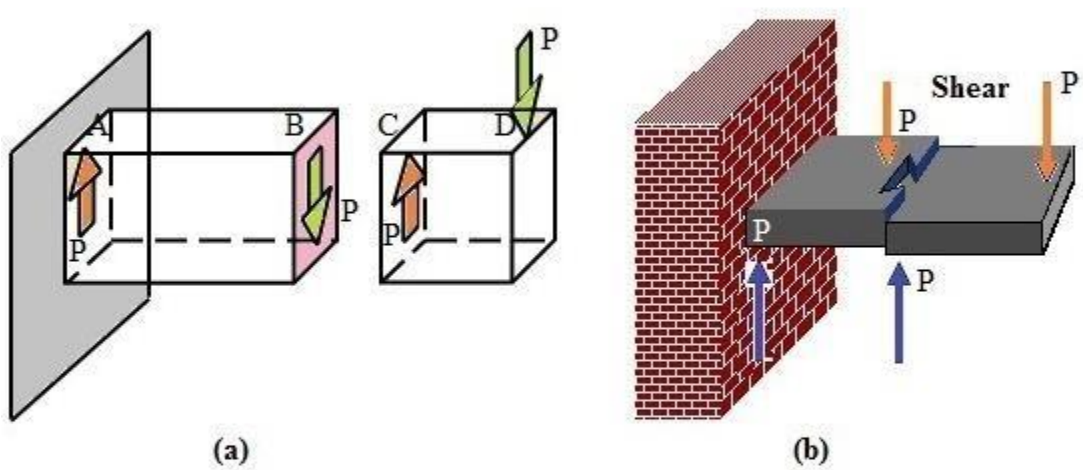


Fig.11 Shearing off beam

Shear force at any section of a beam is defined as the ***net or unbalanced vertical force on either side of the section***. It is the algebraic sum of vertical components of all the forces acting on the beam on either left side or right side of the section. The effect of shear force is to shear off or cut the member at a section. It is similar to the effect of scissor cutting the page of paper.

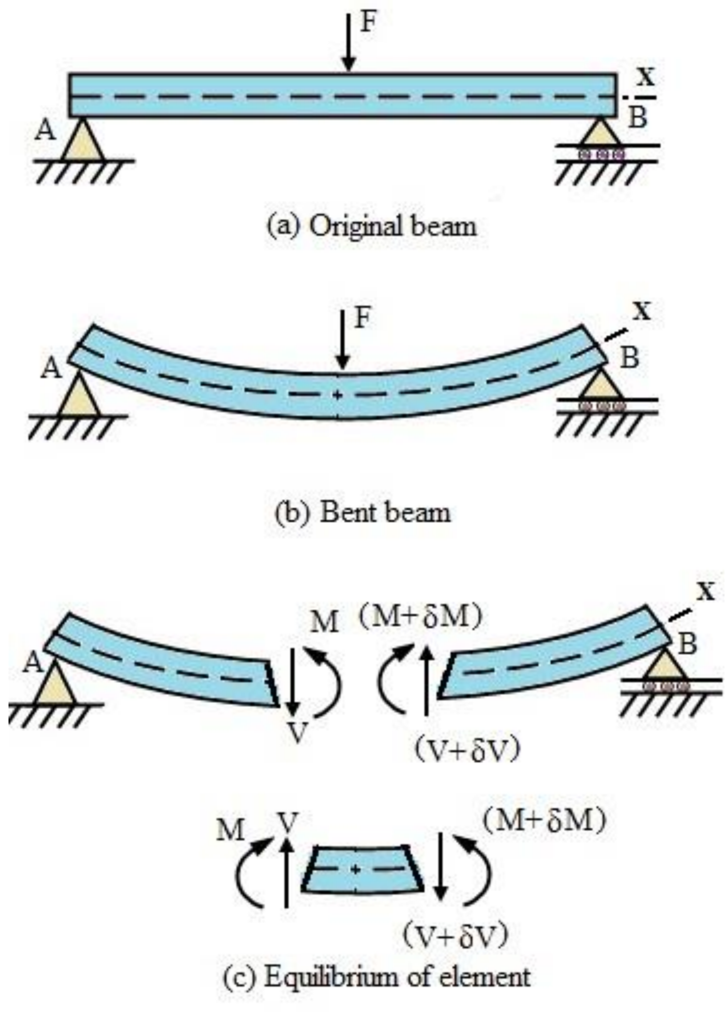


Fig.11 Bending of beam

The moment which tends to bend the beam in plane of load is known as bending moment. In other word bending moment at any section of a beam is the **n*et or unbalanced moment due to all forces on either side of the section***. Bending moment at any section is the algebraic sum of the moments due to all forces acting on the beam on either right or left side of the section. The effect of bending moment is to bend the element.

Sign convention:

The shear force and bending moment are vector quantities and as a matter of convenience are assigned the following sign convention.

Shear force acting in the upward direction to the left hand side of the section and downward direction to the right hand side of the section is considered to be positive & vice-versa.

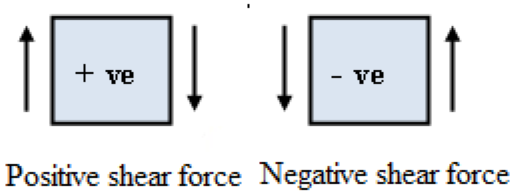


Fig.12

Bending moment is considered to be positive when it is acting in the clockwise direction on the left hand side of the section (L.H.S) (or) when it is acting in the counter-clockwise direction on the right hand side of the section (R.H.S) as the section & vice versa.

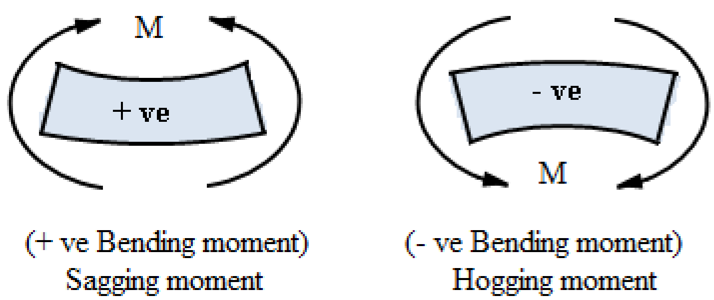


Fig. 13

SHEAR FORCE AND BENDING MOMENT DIAGRAMS:

Graphical representation of variation of shear force along the length of the beam for any given loading condition is known as ***shear force diagram*** (SFD). If x denotes the length of the beam, then shear force 'F' is function of x, i.e. $F(x)$.

Similarly, graphical representation of variation of bending moment along the length of the beam for any given loading condition is known as ***bending moment diagram*** (BMD). If x denotes the length of the beam, then bending moment is function of x, and is denoted as $M(x)$.

Shear force diagram and bending moment diagram are helpful for further analysis and design of beam.

SFD and BMD of a beam reveal the following important information at salient points in the beam. These are maximum shear force, maximum bending moment, point of contralexture or point of inflexion, etc.

RELATIONS BETWEEN LOAD, SHEAR FORCE AND BENDING MOMENT

Consider a beam *AB* carrying generalized loading as shown in the figure. Take an element of infinitesimal length δx between section 1-1 and 2-2 at a distance of x from the left hand support *A*. The free body diagram of the element is drawn with positive sense of the shear forces and bending moments.

The intensity of loading over the length of the element may be taken as constant, i.e., w. Considering equilibrium of the element,

Resolving the forces vertically, $\sum V = 0$

$$F = w\delta x + F + \delta F$$

$$\delta F = -w\delta x$$

$$\frac{\delta F}{\delta x} = -w$$

In the limiting case, as $\delta x \to 0$, $\dfrac{dF}{dx} = -w$ (1)

So, the rate of change of shear force is equal to the intensity or rate of loading.

Taking moments of the forces and couples about the section 2-2, $\sum M_2 = 0$

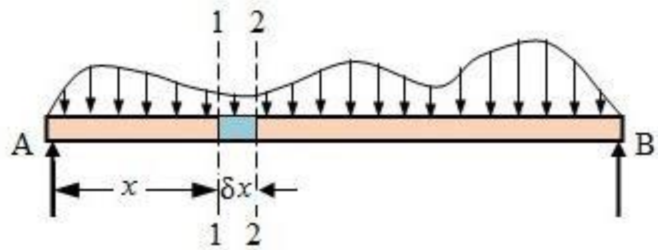
$$M + \delta M + w\frac{(\delta x)^2}{2} = M + F\delta x$$

Neglecting small quantities of higher order, we have

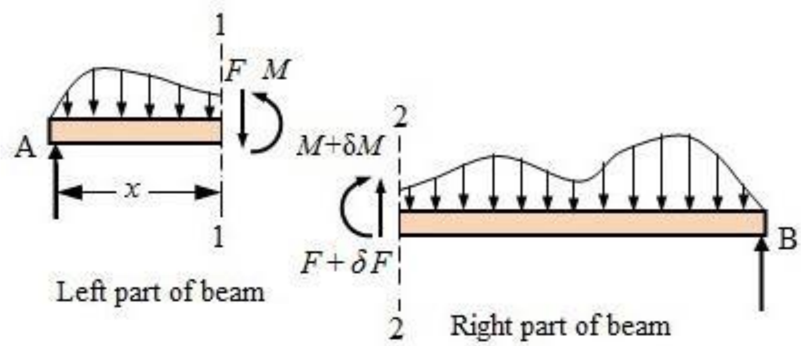
$$\frac{\delta M}{\delta x} = F$$

In the limiting case as $\delta x \to 0$, $\dfrac{dM}{dx} = F$ (2)

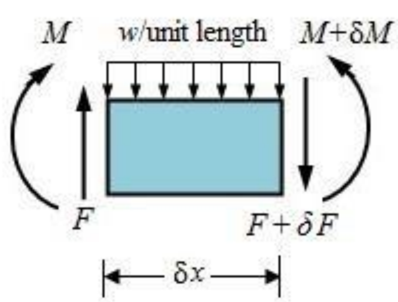
The above equation shows that the rate of change of bending moment is equal to the shear force at the section. Also bending moment would be maximum at a section where shear force is zero.



(a) Loaded beam



(b) Free body diagram of two parts



(c) Enlarged view of element

Fig. 14

Evaluation of Shear Force and Bending Moment:

Thus analysis of beam for shear force and bending moment is carried out by the following process.

1. Determine the reactions at the supports by considering the entire beam as a rigid body and applying equations of equilibrium.
2. Take sections at different points on the beam near supports and load application points.
3. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending moments.
4. Draw shear force and bending moment diagram.
5. Identify the maximum shear and bending-moment from plots of their distributions.
6. Find the position of *point of contaflexure* or *point of inflexion*.

Numerical

1. Draw the Shear force and bending moment diagram for a cantilever beam of length L carrying a point load W at its free end.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0,\ V_A = W\ (\uparrow)$

Taking moment about $A, \sum M_A = 0,\ W \times L + M_A = 0$

$$M_A = -WL \text{ (counter - clockwise)}$$

Calculation of Shear force and bending moments:

Considering from the right hand side B, as the origin, take a section 1-1 at a distance of x from B between B and $A\ (0 \le x \le L)$.

Shear force at 1-1, $F_x = W$

Shear force at B, i.e., $x = 0\ \ F_B = W$

Shear force at A, $i.e, x = L,\ F_A = W$

Bending moment at 1-1, $M_x = -Wx$

Bending moment at B, $i.e, x = 0, M_B = 0$

Bending moment at A, $i.e, x = L, M_A = -WL$

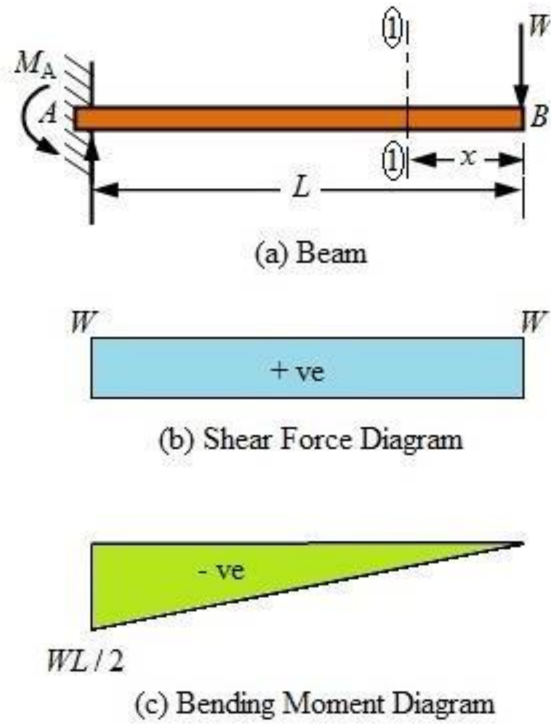


Fig.

2. Draw the Shear force and bending moment diagram for a cantilever beam of length L carrying uniformly distribute load of intensity w over the entire span.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0$, $V_A = wL$

Taking moment about A, $\sum M_A = 0$, $wL \times \frac{L}{2} + M_A = 0$

$$M_A = -\frac{wL^2}{2} \text{ (counter - clockwise)}$$

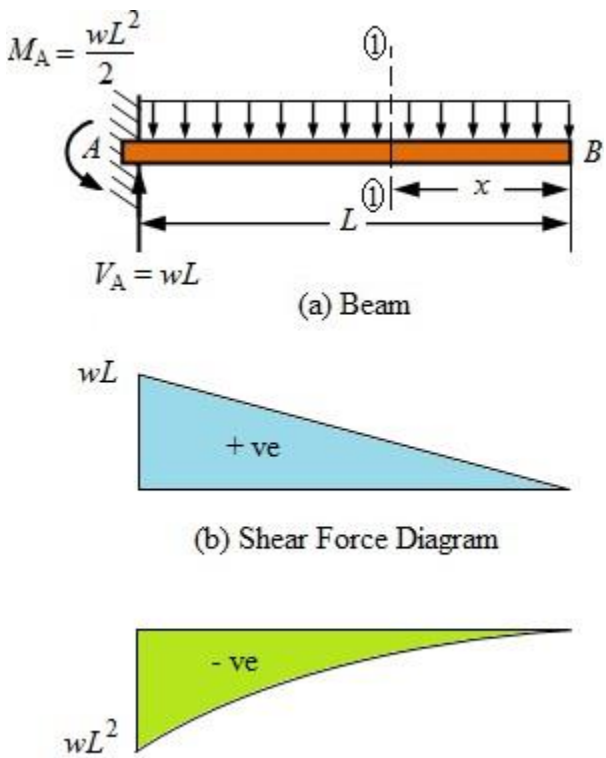
Calculation of Shear force and bending moments:

Considering from the right hand side *B*, as the origin, take a section 1-1 at a distance of *x* from *B* between *B* and *A* $(0 \le x \le L)$.

Shear force at 1-1, $F_x = wx$

Shear force at *B*, i.e., $x = 0$ $F_B = 0$

Shear force at *A*, $i.e, x = L, F_A = wL$



(a) Beam

(b) Shear Force Diagram

(c) Bending Moment Diagram

Bending moment at 1-1, $M_x = -wx \times \left(\frac{x}{2}\right) = -\frac{wx^2}{2}$

Bending moment at *B*, $i.e, x = 0, M_B = 0$

Bending moment at A, $i.e, x = L, M_A = -\frac{wL^2}{2}$

3. Draw the Shear force and bending moment diagram for a cantilever beam of length *L* carrying uniformly distribute load of intensity *w* per unit length from the fixed support to the centre of the beam.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0,\ V_A = \dfrac{wL}{2}$

Taking moment about $A, \sum M_A = 0,\ \dfrac{wL}{2} \times \dfrac{L}{4} + M_A = 0$

$$M_A = -\frac{wL^2}{8} \text{ (counter - clockwise)}$$

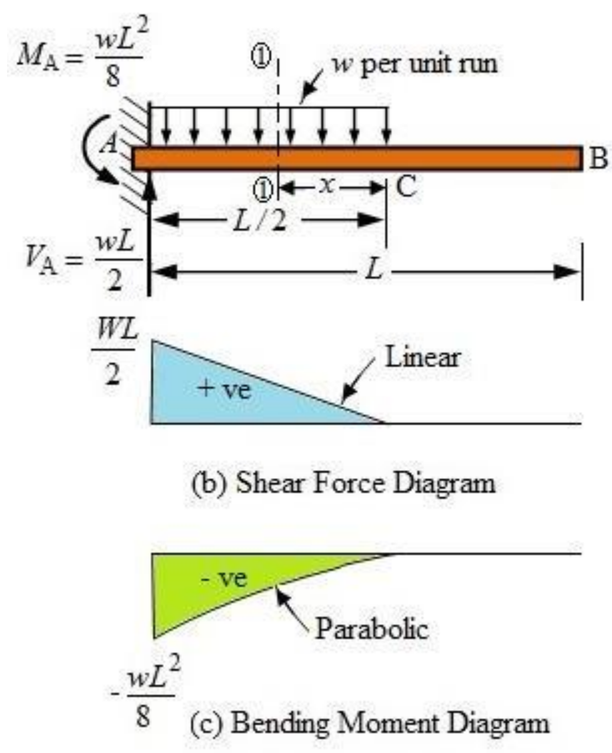


Fig.

Calculation of Shear force and bending moments:

Shear force and bending moment at the free end B, $F_B = 0$; $M_B = 0$

Shear force and bending moment anywhere between B and C is zero since there is no load on the beam in this portion when considered from right side.

Now, considering C as the origin, take a section 1-1 at a distance of x from C between C and A $\left(0 \le x \le \frac{L}{2}\right)$.

Shear force at 1-1, $F_x = wx$

Shear force at A, $i.e, x = L$, $F_A = w\frac{L}{2} = \frac{wL}{2}$

Bending moment at 1-1, $M_x = -wx \times \left(\frac{x}{2}\right) = -\frac{wx^2}{2}$

Bending moment at A, $i.e, x = \frac{L}{2}, M_A = -\frac{w\left(\frac{L}{2}\right)^2}{2}$

$$= -\frac{wL^2}{8}$$

4. Draw the Shear force and bending moment diagram for a cantilever beam of length L carrying uniformly distribute load of intensity w per unit length from the free end up to a distance of a.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Sum of the vertical forces, $\sum V = 0$, $V_A = wa$

Taking moment about A, $\sum M_A = 0$, $wa \times \left(L - \frac{a}{2}\right) + M_A = 0$

$$M_A = -\frac{wa}{2}(L - 2a) \quad \text{(counter - clockwise)}$$

Calculation of Shear force and bending moments:

Shear force and bending moment at the free end B, $F_B = 0$; $M_B = 0$

Now, considering B as the origin, take a section 1-1 at a distance of x from B between B and C $(0 \le x \le a)$.

Shear force at 1-1, $F_x = wx$

Shear force at B, $i.e, x = 0$, $F_C = w \times 0 = 0$

Shear force at C, $i.e, x = a$, $F_C = wa$

Bending moment at 1-1, $M_x = -wx \times \left(\dfrac{x}{2}\right) = -\dfrac{wx^2}{2}$

Bending moment at C, $i.e, x = a, M_A = -\dfrac{wa^2}{2}$

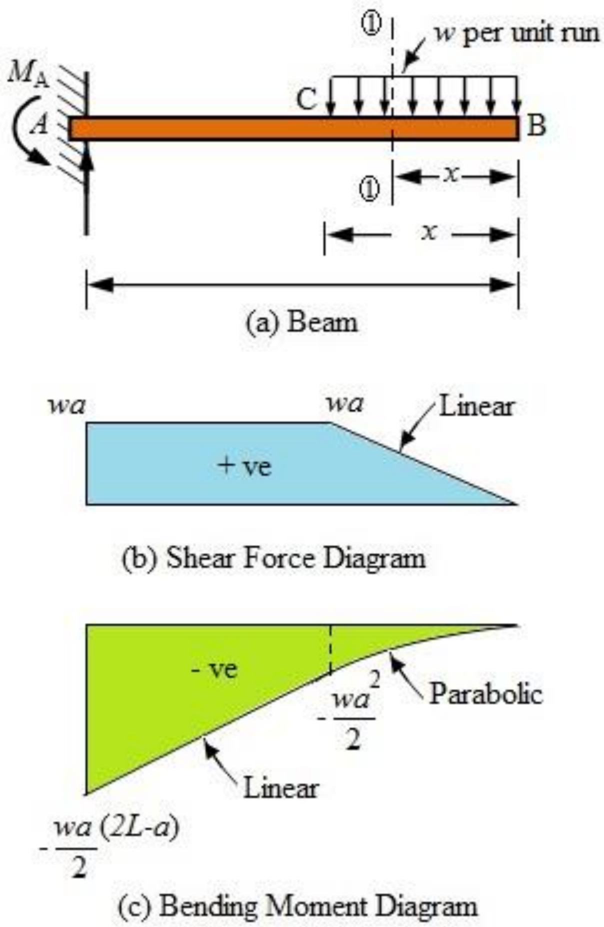


Fig.

Now, take a section 2-2 at a distance of x from B between C and A $(a \le x \le L)$.

Shear force at 2-2, $F_x = F_C = wa$

Shear force will remain same as wa from C to A.

Bending moment at 2-2, $M_x = -wa \times \left(x - \dfrac{a}{2}\right)$

Bending moment at A, *i.e.*, $x = L$, $M_x = -wa\left(L - \frac{a}{2}\right) = -\frac{wa}{2}(2L - a)$

5. A cantilever of 3.5 m long carries point loads of 15 kN, 15 kN and 7.5 kN at 1 m, 1 m and 1.5 m respectively from the fixed end. Draw the Shear force and bending moment diagram for the beam.

Solution: Calculation of Shear force and bending moments:

Portion *BD*: At section 1-1 at a distance x from B between B and D $(0 \le x \le 1.5m)$

Shear force at 1-1, $F_x = 7.5kN$ (constant from B to just right of D)

Shear force at B, $F_B = 7.5\ kN$

Shear force just right of D, $F_{DL} = 7.5\ kN$

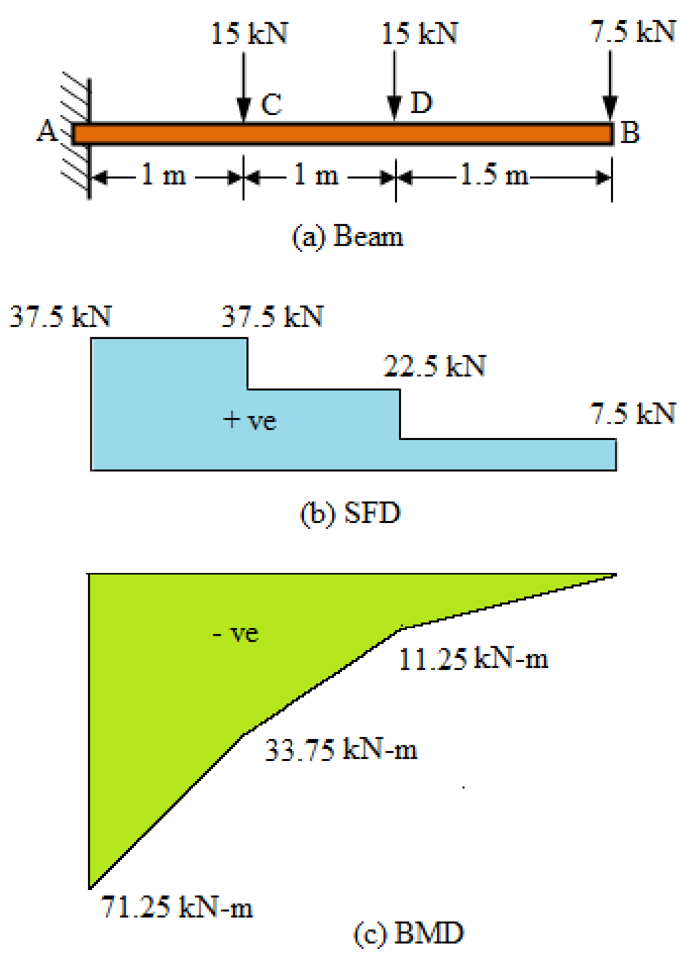


Fig.

Bending moment at 1-1, $M_x = -7.5x$

Bending moment at B, $i.e., at\ x = 0, M_B = -7.5 \times 0 = 0$

Bending moment at D, $i.e., at\ x = 1.5, M_D = -7.5 \times 1.5 = 11.25 kN-m$

Portion *DC*: At section 2-2 at a distance x from B between D and C $(1.5 \leq x \leq 2.5m)$

Shear force at 2-2, $F_x = 7.5 + 15$ (constant from D to just right of C)

Shear force at D, $F_D = 22.5\ kN$

Shear force just right of C, $F_{CL} = 22.5\ kN$

Bending moment at 2-2, $M_x = -7.5x - 15(x - 1.5)$

$$= -22.5x + 22.5$$

Bending moment at C, $i.e., at\ x = 2.5, M_B = -22.5 \times 2.5 + 22.5 = -33.75 kN - m$

Portion *CA*: At section 3-3 at a distance x from B between C and A $(2.5 \leq x \leq 3.5m)$

Shear force at 3-3, $F_x = 7.5 + 15 + 15$ (constant from C to A)

Shear force at C, $F_C = 37.5\ kN$

Shear force at A, $F_A = 37.5\ kN$

Bending moment at 3-3, $M_x = -7.5x - 15(x - 1.5) - 15(x - 2.5)$

$$= -37.5x + 60$$

Bending moment at A, $i.e., at\ x = 3.5m, M_B = -37.5 \times 3.5 + 60 = -71.25 kN - m$

6. A cantilever of 1.6 *m* long carries a uniformly distributed load of intensity 1.5 *kN/m* over the entire span and a point load of 2.5 *kN* at the free end. Draw the Shear force and bending moment diagram for the beam.

Solution:

Calculation of Shear force and bending moments:

Considering from the right hand side B, as the origin, take a section 1-1 at a distance of x from B between B and A $(0 \leq x \leq 1.6m)$.

Shear force at 1-1, $F_x = 2.5 + 1.5x$

Shear force at B, $i.e., x = 0, F_B = 2.5 + 1.5 \times 0 = 2.5 kN$

Shear force at A, $i.e., x = 1.6m, F_x = 2.5 + 1.5 \times 1.6 = 4.9 kN$

Bending moment at 1-1, $M_x = -2.5x - 1.5x\left(\frac{x}{2}\right)$

$$= -2.5x - 0.75x^2$$

Bending moment at B, $i.e., x = 0, M_B = 0$

Bending moment at A, $i.e., x = 1.6, M_B = -2.5 \times 1.6 - 0.75 \times 1.6^2$

$$= -5.92\, kN - m$$

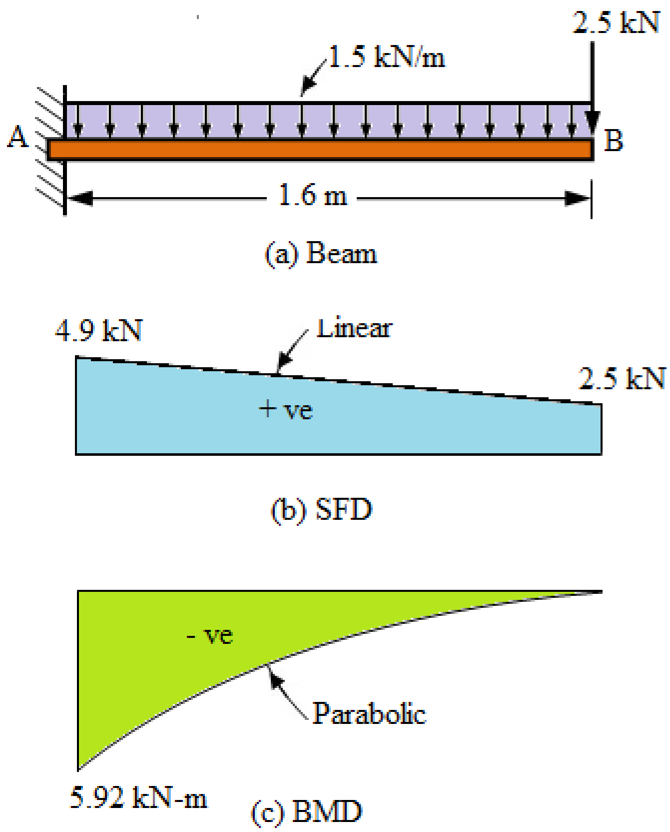


Fig.

7. A cantilever of 1.5 *m* long is loaded with a uniformly distributed load of intensity 2 *kN/m* and a point load of 2.5 *kN* as shown in the figure. Draw the Shear force and bending moment diagram for the cantilever.

Calculation of Shear force and bending moments:

Considering from the right hand side B, as the origin, take a section 1-1 at a distance of x from B between B and $D\,(0 \le x \le 0.25m)$.

Shear force at 1-1, $F_x = 2x$

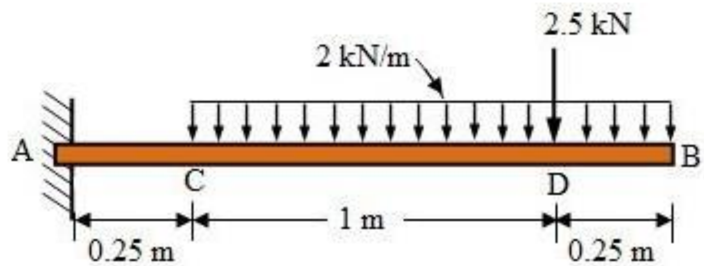
Shear force at B, $i.e., x=0, F_B = 2\times 0 = 0$

Shear force just right of D, $i.e., x = 0.25m, F_D = 2\times 0.25 = 0.5kN$

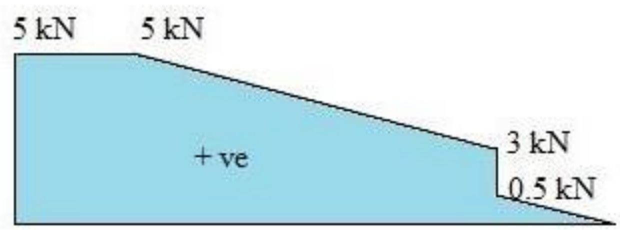
Shear force at D, $i.e., x = 0.25m, F_D = 0.5+2.5 = 3kN$

Bending moment at 1-1, $M_x = -2x\left(\frac{x}{2}\right) = -x^2$

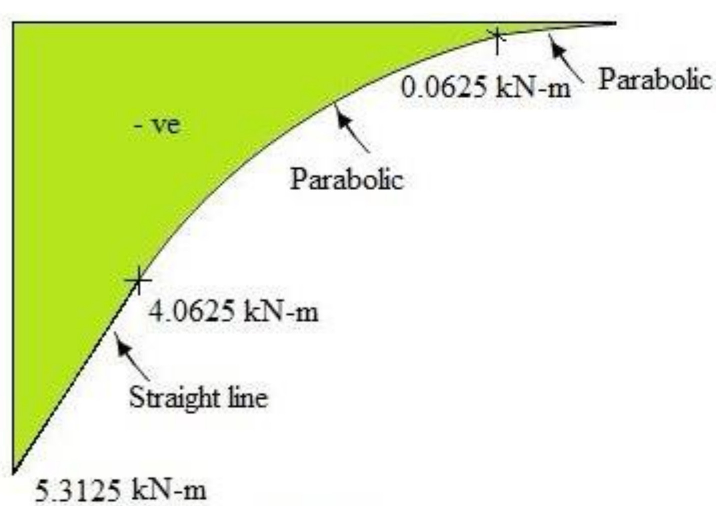
Bending moment at D, i.e., $x = 0.25$, $M_D = -(0.25)^2 = -0.0625\,kN-m$



(a) Beam



(b) SFD



(c) BMD

Fig.

Now, take a section 2-2 at a distance of x from B between D and $C\,(0.25 \le x \le 1.25m)$.

Shear force at 1-1, $F_x = 2x + 2.5$

Shear force at D, $i.e., x = 0.25, F_B = 2 \times 0.25 + 2.5 = 3kN$

Shear force C, $i.e., x = 1.25m, F_D = 2 \times 1.25 + 2.5 = 5kN$

Bending moment at 1-1, $M_x = -2x\left(\frac{x}{2}\right) - 2.5(x - 0.25)$

$$= -x^2 - 2.5x + 0.625$$

Bending moment at C, i.e., x = 1.25, $M_C = -1.25^2 - 2.5 \times 1.25 + 0.625$

$$= -4.0625\, kN - m$$

8. Calculate the shear force and bending moment for the beam subjected to a concentrated load of W as shown in the figure. Draw the shear force diagram (SFD) and bending moment diagram (BMD).

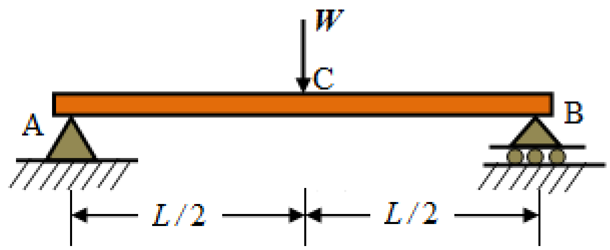


Fig.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Taking moment about B, $\sum M_B = 0$, $V_A \times L = W \times \frac{L}{2}$

$$V_A = \frac{W}{2}$$

Sum of the vertical forces, $\sum V = 0$, $V_A + V_B = W$

Hence, $$V_B = \frac{W}{2}$$

Calculation of Shear force and bending moments:

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $\left(0 < x < L/2\right)$.

Shear force at 1-1, $F_x = V_A = \frac{W}{2}$

Shear force at A, i.e., $x = 0$ $F_A = \frac{W}{2}$

Shear force just left of C, i.e., $x = 0$ $i.e., x = \frac{L}{2}$, $F_{LC} = \frac{W}{2}$

Bending moment at 1-1, $M_x = V_A \times x = \frac{W}{2} x$

Bending moment at A, i.e., $x = 0$, $M_A = \frac{Wx}{2} = 0$

Bending moment at C, i.e., $x = \frac{L}{2}$, $M_C = \frac{W}{2} \times \frac{L}{2} = \frac{WL}{4}$

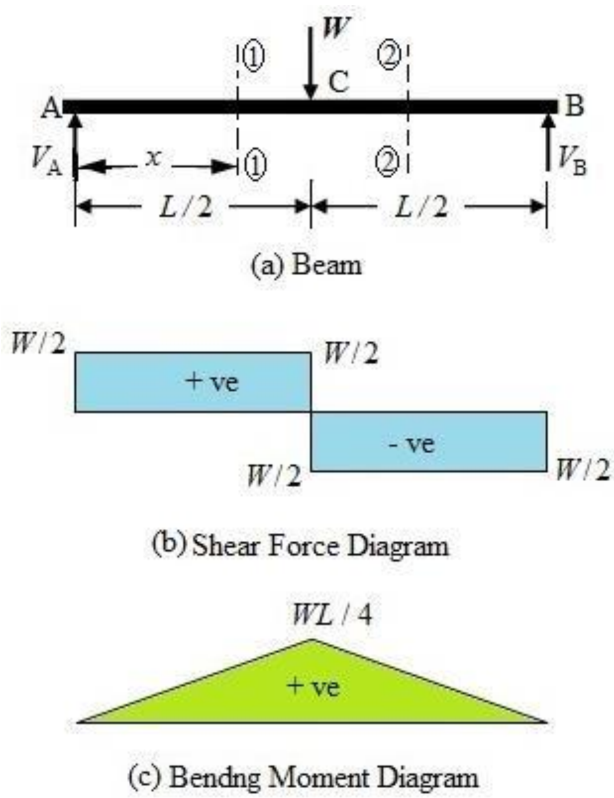


Fig.

Take a section 2-2 at a distance of x from A between C and $B\left(\frac{L}{2} \le x \le L\right)$.

Shear force at 2-2, $F_x = \frac{W}{2} - W = -\frac{W}{2}$

Shear force at $C\left(x = \frac{L}{2}\right)$, $F_C = \frac{-W}{2}$

Shear force at B, $(x = L)$, $F_B = \frac{-W}{2}$

Bending moment at 2-2, $M_x = \frac{W}{2}x - W\left(x - \frac{L}{2}\right)$

$$= \frac{Wx}{2} - Wx + \frac{WL}{2}$$

$$= -\frac{Wx}{2} + \frac{WL}{2}$$

Bending moment at B, i.e., $x = L$, $M_B = -\frac{Wl}{2} + \frac{WL}{2} = 0$

Bending moment at C, i.e., $x = \frac{L}{2}$, $M_C = -\frac{W}{2}\left(\frac{L}{2}\right) + \frac{WL}{2} = \frac{WL}{4}$

9. Draw the Shear force and bending moment diagram for a simply supported beam of length L carrying uniformly distribute load of intensity w per unit length over the entire span.

Solution:

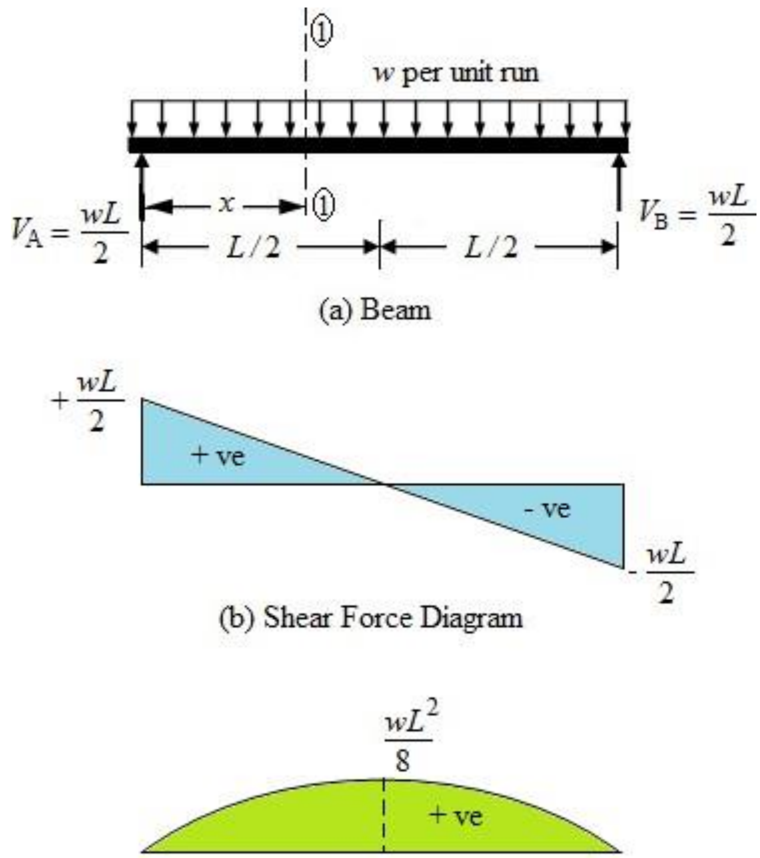
Evaluation of support reactions:

The simply supported beam with uniformly distributed load over the entire span is symmetrically loaded symmetric beam. Hence, reactions at both supports are equal.

$$R_A = R_B = \frac{wL}{2}$$

Calculation of Shear force and bending moments:

In a symmetric beam, we need only to analyze half of the beam for shear force and bending moment. The other half will just be the mirror-image of the first half.



(a) Beam

(b) Shear Force Diagram

(c) Bending Moment Diagram

Fig.

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $\left(0 \le x \le L/2\right)$.

Shear force at 1-1, $V_x = V_A - wx = \frac{wL}{2} - wx$

Shear force at A, i.e., $x = 0$ $F_A = \frac{wl}{2}$

Shear force at C, i.e., $F_C = \frac{wL}{2} - w \times \frac{L}{2} = 0$

Bending moment at 1-1, $M_x = V_A \times x - wx\left(\frac{x}{2}\right) = \frac{wL}{2}x - wx\left(\frac{x}{2}\right)$

$$= \frac{wL}{2}x - \frac{wx^2}{2}$$

Bending moment at A, i.e., $x = 0$ $M_x = 0$

Bending moment at C, $M_C = \frac{wL}{2}\left(\frac{L}{2}\right) - \frac{w\left(\frac{L}{2}\right)^2}{2} = \frac{wL^2}{4} - \frac{wL^2}{8}$

$= \frac{wL^2}{8}$

Bending moment equation is a quadratic in form, hence the bending moment diagram will be parabolic between A and B.

Due to symmetry, the bending moment and shear force for the other half at respective point of symmetry will be same as the first half AB.

10. A simply supported beam shown in the figure carries two concentrated loads and a uniformly distribute load. Analyze the beam for shear force and bending moment, and draw the SFD and BMD.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Taking moment about B, $\sum M_B = 0$, $V_A \times 8 = 25\times 6 + 15\times 4 + 7.5\times 4\times 2$

$V_A = 33.75kN$

Sum of the vertical forces, $\sum V = 0$, $33.75 + V_B = 25 + 15 + 7.5\times 4$

Hence, $V_B = 70 - 33.75 = 36.25kN$

Calculation of Shear force and bending moments:

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $(0 \le x \le 2)$.

Shear force at 1-1, $F_x = V_A = 33.75\,kN$

Shear force at A, i.e., $x = 0$ $F_A = 33.75kN$

Shear force just left of C, $i.e., x = 2$, $F_{LC} = 33.75kN$

Shear force at C, $i.e., x = 2$, $F_C = 33.75 - 25 = 8.75\,kN$

Bending moment at 1-1, $M_x = V_A \times x = 33.75x$

Bending moment at C, $i.e., x = 2$, $M_C = 33.75 \times 2 = 67.5\,kN - m$

Take a section 2-2 at a distance of x from A between C and D $(2 \le x \le 4)$.

Shear force at 2-2, $F_x = 33.75 - 25 = 8.75\,kN$

Shear force at C, i.e., $x = 2$, $F_C = 8.75\,kN$

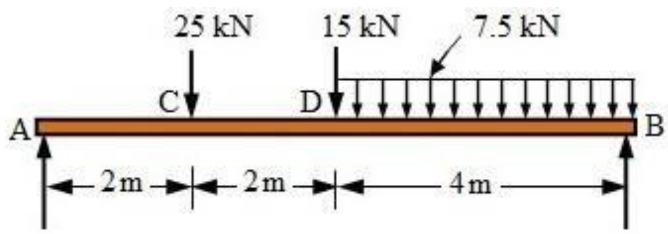
Shear force just left of D, $i.e., x = 4$, $F_{LD} = 8.75kN$

Shear force at D, $i.e., x = 4$, $F_D = 8.75 - 15 = -6.25kN$

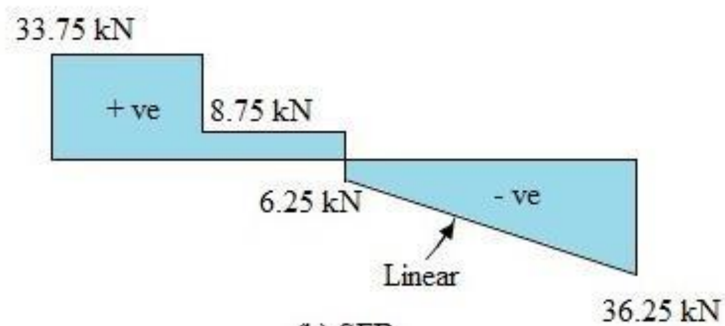
Bending moment at 2-2, $M_x = 33.75x - 25(x - 2)$

$$M_x = 8.75x + 50$$

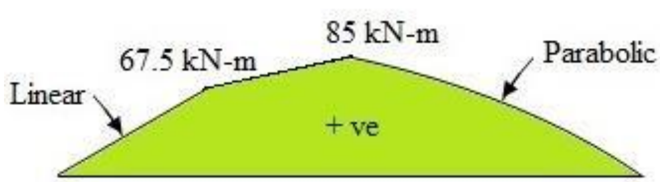
Bending moment at C, $i.e., x = 4$, $M_D = 8.75 \times 4 + 50 = 85kN - m$



(a) Beam



(b) SFD



(c) BMD

Fig.

Now, considering from the right side and taking B as the origin, take a section 3-3 at a distance of x from B between B and D $(0 \le x \le 4)$.

Shear force at 3-3, $F_x = -36.25 + 7.5x$

Shear force at B, i.e., $x = 0$ $F_B = 36.25 kN$

Shear force just right of D, i.e., at $x = 4$, $F_x = -36.25 + 7.5 \times 4 = -6.25 kN$

Shear force at D, $i.e., x = 4, F_D = -6.25 + 15 = 8.75 kN$

Bending moment at 3-3, $M_x = 36.25x - \frac{7.5}{2}x^2 = 36.25x - 3.75x^2$

Bending moment at D, i.e., $x = 4$, $M_D = 36.25 \times 4 - 3.75 \times 4^2 = 85 kN - m$

11. Draw the shear force and bending moment diagram for the overhanging beam shown in the figure.

Solution:

Evaluation of support reactions:

Considering the equilibrium of the beam and applying static equations of equilibrium,

Taking moment about A, $\sum M_A = 0$, $V_D \times 4 = 20 \times 5 + 50 \times 2 + 20 \times 2 \times 1$

$$V_D = 60 kN$$

Sum of the vertical forces, $\sum V = 0$, $V_A + 60 = 20 \times 2 + 50 + 20$

Hence, $V_A = 110 - 60 = 50 kN$

Calculation of Shear force and bending moments:

Considering A as the origin, take a section 1-1 at a distance of x from A between A and C $(0 \le x \le 2)$.

Shear force at 1-1, $F_x = 50 - 20x$

Shear force at A, i.e., $x = 0$ $F_A = 50 kN$

Shear force just left of C, $i.e., x = 2, F_{LC} = 50 - 20 \times 2$

$$F_{LC} = 10 kN$$

Shear force at C, $i.e., x = 2, F_C = 10 - 50 = -40 kN$

Bending moment at 1-1, $M_x = V_A \times x - 20 \times \frac{x^2}{2}$

$$= 50x - 10x^2$$

Bending moment at C, *i.e.*, $x = 2$, $M_C = 50 \times 2 - 10 \times 2^2 = 60\,kN - m$

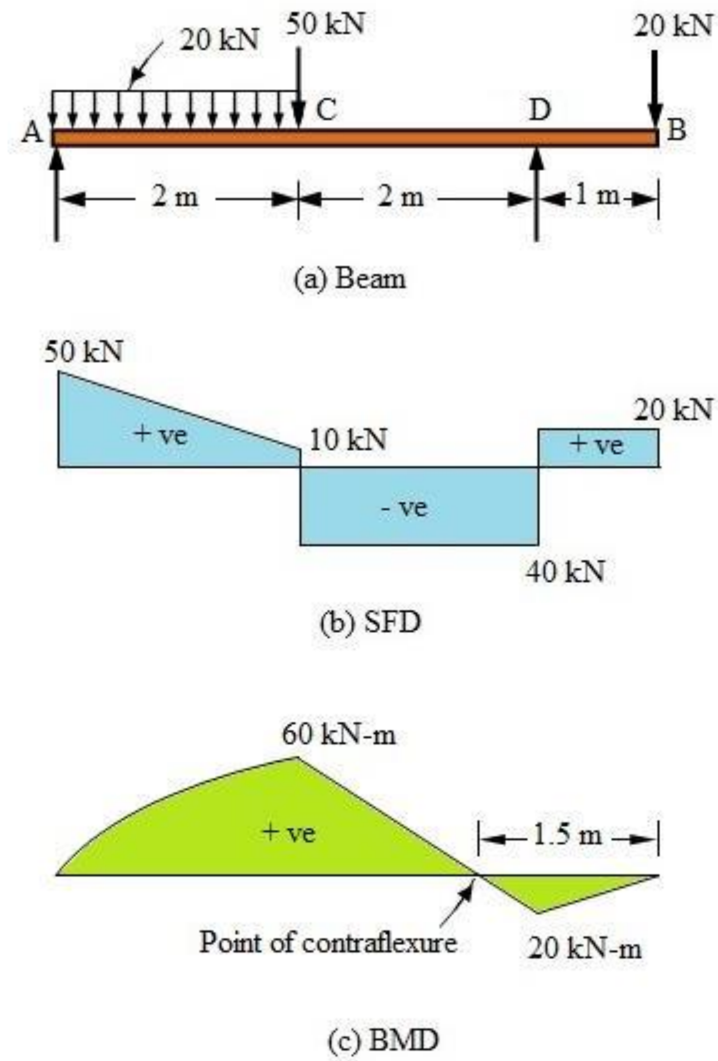


Fig.

Now, considering from the right side and taking B as the origin, take a section 2-2 at a distance of x from B between B and D $(0 \le x \le 4)$.

Shear force at 2-2, $F_x = 20\,kN$

Shear force at B, i.e., $x = 0$ $F_B = 20\,kN$

Shear force just right of D, i.e., at $x = 1$, $F_D = 20kN$

Shear force at D, $i.e., x = 1$, $F_D = 20 - 60 = -40\,kN$

Bending moment at 2-2, $M_x = -20x$

Bending moment at D, i.e., $x = 1$, $M_B = 0$

Bending moment at D, $i.e., M_D = -20 \times 1 = -20\,kN - m$

Take a section 3-3 at a distance of x from B between D and C $(1 \le x \le 3)$.

Shear force at 3-3, $F_x = 20 - 60 = -40\,kN$

Shear force at D, i.e., $x = 1$, $F_D = -40\,kN$

Bending moment at 3-3, $M_x = -20x + 60(x-1)$

$$= 40x - 60$$

Bending moment at C, i.e., $x = 3$, $M_x = 40 \times 3 - 60 = 60\,kN - m$

It is observed that bending moment changes sign between D and C. So, point of contraflexure exists between D and C.

Equating bending moment equation to zero, we get

$$40x - 60 = 0$$

$$x = 1.5\,m$$

Point of contraflexure:

A point of contraflexure is a point where the curvature of the beam changes signs. It is sometimes referred to as a ***point of inflexion***. In other words, point of contraflexure is a point where bending moment changes its sign from positive to negative or from negative to positive through zero. This means, bending moment is zero at point of contraflexure.

Columns and Struts

- Any member subjected to axial compressive load is called a column or Strut.
- A vertical member subjected to axial compressive load – *COLUMN* (Eg: Pillars of a building)
- An inclined member subjected to axial compressive load - *STRUT*
- A strut may also be a horizontal member
- Load carrying capacity of a compression member depends not only on its cross sectional area, but also on its length and the manner in which the ends of a column are held.
- Equilibrium of a column – Stable, Unstable, Neutral.
- Critical or Crippling or Buckling load – Load at which buckling starts
- Column is said to have developed an elastic instability.

Classification of Columns

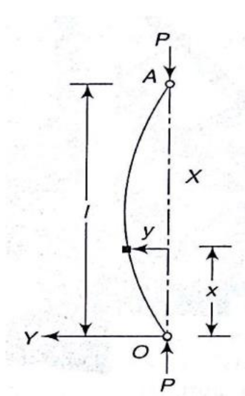
- According to nature of failure – short, medium and long columns
- 1. Short column – whose length is so related to its c/s area that *failure occurs mainly due to direct compressive stress* only and the role of bending stress is negligible
- 2. Medium Column - whose length is so related to its c/s area that *failure occurs by a combination of direct compressive stress and bending stress*
- 3. Long Column - whose length is so related to its c/s area that *failure occurs mainly due to bending stress* and the role of direct compressive stress is negligible

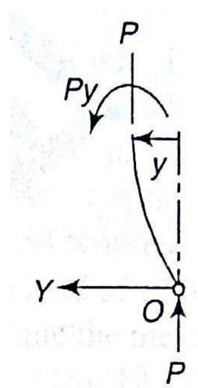
Euler's Theory

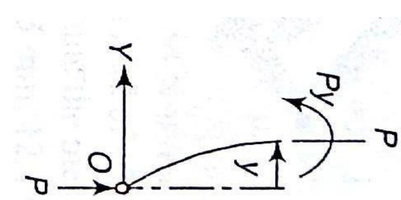
- Columns and struts which fail by buckling may be analyzed by Euler's theory
- <u>Assumptions made</u>

- the column is initially straight
- the cross-section is uniform throughout
- the line of thrust coincides exactly with the axis of the column
- the material is homogeneous and isotropic
- the shortening of column due to axial compression is negligible.

Case (i) Both Ends Hinged







$$EI\,\frac{d^2y}{dx^2} = M = -Py$$

$$EI\frac{d^2y}{dx^2} = M = -Py$$

The equation can be written as $\frac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \frac{P}{EI}$

The solution is $y = A\sin\alpha x + B\cos\alpha x$

At $x = 0,\ y = 0,\ \therefore\ B = 0$

at $x = l,\ y = 0$ and thus $A\sin\alpha l = 0$

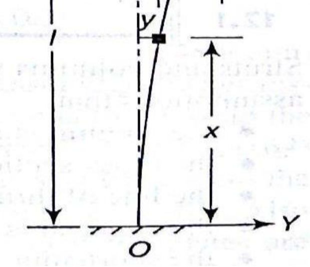
If $A = 0$, y is zero for all values of load and there is no bending.

$\therefore\ \sin\alpha l = 0$ or $\alpha l = \pi$ (considering the least value)

or $\alpha = \pi / l$

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \frac{\pi^2 EI}{l^2}$

Case (ii) One end fixed other free



$$EI\frac{d^2y}{dx^2} = M = P(a-y) = Pa - Py$$

$$EI\frac{d^2y}{dx^2} = M = P(a-y) = Pa - Py$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{P \cdot a}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{P.a}{EI\alpha^2}$

$$= A\sin\alpha x + B\cos\alpha x + a$$

$$x = 0, y = 0, \;\therefore B = -a;$$

$$x = 0, \; \frac{dy}{dx} = 0$$

or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0 \quad$ or $\quad A = 0$

$$y = -a\cos\alpha x + a = a(1-\cos\alpha x)$$

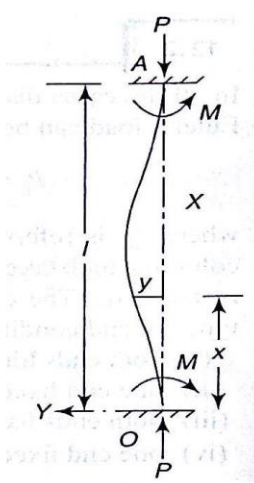
$$\text{At } x = l, y = a, \therefore a = a(1 - \cos \alpha l)$$

$$\text{or } \cos \alpha l = 0 \quad \text{or} \quad \alpha l = \frac{\pi}{2} \text{ (considering the least value)}$$

$$\alpha = \pi / 2l$$

$$\therefore \text{Euler crippling load, } P_e = \alpha^2 EI = \frac{\pi^2 EI}{4l^2}$$

Case (iii) Fixed at both ends



$$EI \frac{d^2 y}{dx^2} = -Py + M$$

$$EI\frac{d^2y}{dx^2} = -Py + M$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{M}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \frac{M}{EI\alpha^2} = A\sin\alpha x + B\cos\alpha x + \frac{M}{P}$

$x = 0, y = 0, \therefore B = -\frac{M}{P};$

$x = 0, \frac{dy}{dx} = 0$

or $A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0$ or $A = 0$

$$\therefore y = -\frac{M}{P}\cos\alpha x + \frac{M}{P} = \frac{M}{P}(1-\cos\alpha x)$$

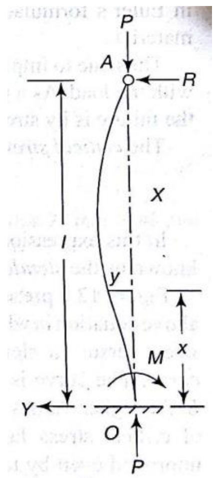
At $x = l, y = 0, \therefore 0 = \frac{M}{P}(1-\cos\alpha l)$ or $\cos\alpha l = 1$

or $\alpha l = 2\pi$ (considering the least value)

or $\alpha = 2\pi/l$

$\therefore$ **Euler crippling load,** $P_e = \alpha^2 EI = \frac{4\pi^2 EI}{l^2}$

Case (iv) One end fixed, other hinged



$$EI\frac{d^2y}{dx^2} = -Py + R(l-x)$$

$$EI\frac{d^2y}{dx^2} = -Py + R(l-x)$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{R(l-x)}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{R(l-x)}{EI\alpha^2}$

$$= A\sin\alpha x + B\cos\alpha x + \frac{R}{P}(l-x)$$

At $x = 0, y = 0, \therefore\ B = -\dfrac{Rl}{P}$;

At $x = 0, \dfrac{dy}{dx} = 0$

or $A\alpha\cos\alpha x - B\alpha\sin\alpha x - \dfrac{R}{P} = 0$

or $A = \dfrac{R}{P\alpha}$

$$\therefore \quad y = \frac{R}{P\alpha}\sin \alpha x - \frac{Rl}{P}\cos \alpha x + \frac{R}{P}(l - x)$$

$$\text{At } x = l, y = 0, \therefore \ 0 = \frac{R}{P\alpha}\sin \alpha l - \frac{Rl}{P}\cos \alpha l$$

or $\quad \tan \alpha l = \alpha l$

$\alpha l = 4.49$ rad (considering the least value)
$\alpha = 4.49 / l$

$$\therefore \text{ Euler crippling load, } P_e = \alpha^2 EI = \frac{4.49^2 EI}{l^2} = \frac{20.2EI}{l^2} \approx \frac{2\pi^2 EI}{l^2}$$

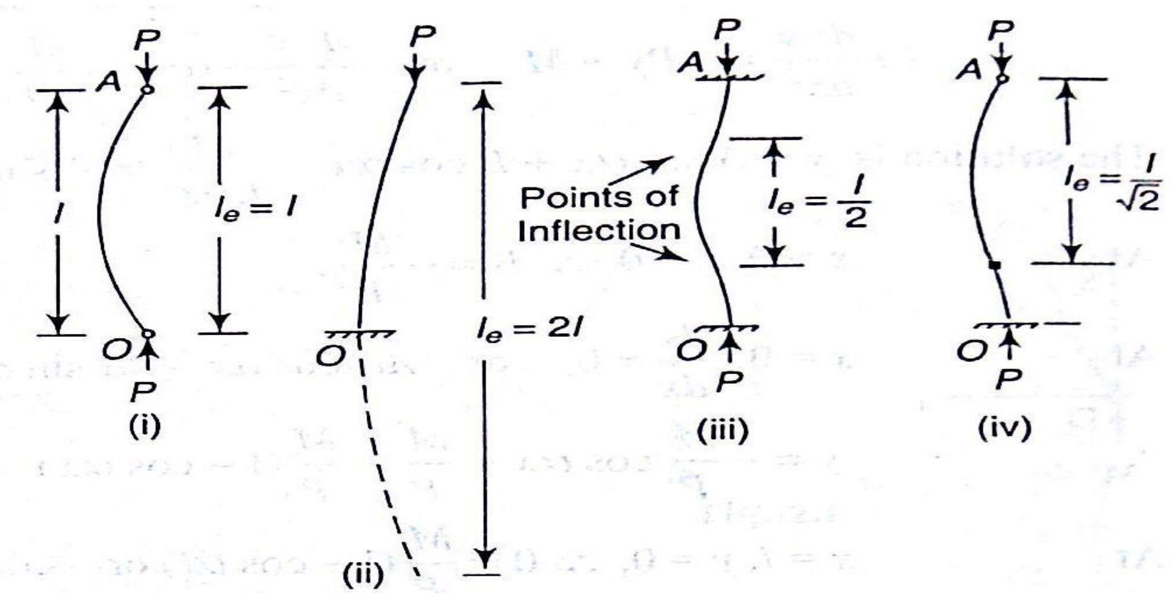
Equivalent Length (l_e)

Euler's load can be expressed as $P_e = \frac{\pi^2 EI}{l_e^2}$

where l_e^2 is referred as *equivalent length* of the column which takes into account the type of fixing of the ends.

The equivalent lengths for different types of end conditions are

(i) both ends hinged, $l_e = l$
(ii) one end fixed and the other free, $l_e = 2l$
(iii) both ends fixed, $l_e = l/2$
(iv) one end fixed, other hinged, $l_e = l/\sqrt{2}$



Limitations of Euler's Formula

Assumption – Struts are initially perfectly straight and the load is exactly axial.

There is always some eccentricity and initial curvature present.

In practice a strut suffers a deflection before the Crippling load.

Critical stress (σ_c) – average stress over the cross section

$$\sigma_c = \frac{P_e}{A} = \frac{\pi^2 EI}{Al_e^2}$$

$$= \frac{\pi^2 EAk^2}{Al_e^2}$$

$$\sigma_c = \frac{\pi^2 E}{(l_e/k)^2}$$

- *l*/k is known as **Slenderness Ratio**

Slenderness Ratio

Slenderness ratio is the ratio of the length of a column and the radius of gyration of its cross section

Slenderness Ratio = *l*/k

The Radius of Gyration $\mathbf{k_x}$ of an Area (A) about an axis (x) is defined as:

$$I_x = k_x^2 A$$

$$\mathbf{k_x} = \sqrt{\frac{I_x}{A}}$$

Rankine's Formula OR Rankine-Gorden Formula

- Euler's formula is applicable to long columns only for which *l*/k ratio is larger than a particular value.
- Also doesn't take in to account the direct compressive stress.
- Thus for columns of medium length it doesn't provide accurate results.
- Rankine forwarded an empirical relation

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

where P = Rankine's crippling load
P_c = ultimate load for a strut = $\sigma_u \cdot A$, constant for a material
P_e = Eulerial load for a strut = $\pi^2 EI/l^2$

- For short columns, P_e is very large and therefore $1/P_e$ is small in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_c
- For long columns, P_e is very small and therefore $1/P_e$ is quite large in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_e

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \dfrac{\sigma_c A \cdot l^2}{\pi^2 EI}}$$

$$= \frac{\sigma_c \cdot A}{1 + \dfrac{\sigma_c \cdot A \cdot l^2}{\pi^2 EAk^2}}$$

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\dfrac{l}{k}\right)^2}$$

where σ_c is the crushing stress

a is the Rankine's constant $(\sigma_c/\pi^2 E)$

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EI}}$$

$$= \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c \cdot A \cdot l^2}{\pi^2 EAk^2}}$$

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l}{k}\right)^2}$$

where σ_c is the crushing stress

a is the Rankine's constant $(\sigma_c / \pi^2 E)$

A Factor of Safety may be considered for the value of σ_c in the above formula

Rankine's formula for columns with other end conditions

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2}$$